THREE DIMENSIONAL MODELLING OF ELECTRICAL IMPEDANCE TOMOGRAPHY

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### Abbreviations and notations

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<td>BEM</td>
<td>Boundary Element Method</td>
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<td>ECT</td>
<td>Emission Computed Tomography</td>
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<td>EIT</td>
<td>Electrical Impedance Tomography</td>
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<td>ERT</td>
<td>Electrical Resistivity Tomography</td>
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<tr>
<td>FEM</td>
<td>Finite Element Method</td>
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<td>FP</td>
<td>Forward Problem</td>
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<tr>
<td>IP</td>
<td>Inverse Problem</td>
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<td>IPT</td>
<td>Industrial Process Tomography</td>
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<td>MEIT</td>
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<td>MRI</td>
<td>Magnetic Resonance Tomography</td>
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<td>NMR</td>
<td>Nuclear Magnetic Resonance</td>
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<td>PET</td>
<td>Positron Emission Tomography</td>
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<td>SPECT</td>
<td>Single-Photon Emission Computed Tomography</td>
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<td>SVD</td>
<td>Singular Value Decomposition</td>
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<td>TCT</td>
<td>Transmission Computed Tomography</td>
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<td>X-ray CT</td>
<td>X-ray Computerised Tomography</td>
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<tr>
<td>$\sigma$</td>
<td>Conductivity</td>
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<td>$\epsilon$</td>
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<td>$\mu$</td>
<td>Permeability</td>
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<td>$J$</td>
<td>Current density</td>
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<td>$I$</td>
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<tr>
<td>$i$</td>
<td>Imaginary</td>
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\( \Omega \) - Closed region

\( \partial \Omega \) - Boundary of the region

\( \delta_{0n} \) - Kronecker symbol

\( \delta \) - Delta function

\( k \) - Wavenumber

\( W_i \) - Width of the \( i^{th} \) electrode

\( S_i \) - Height of the \( i^{th} \) electrode

\( Z_i \) - Position of the \( i^{th} \) electrode along the \( Z \) axis

\( \Delta_i \) - Subtentuated angle of the \( i^{th} \) electrode

\( \rho, \phi, z \) - Position (in polar coordinates) at which a solution is sought

\( \rho_0, \phi_0, z_0 \) - Current source location in polar coordinates

\( a \) - Radius of the circular cylinder

\( c \) - Half-length of the circular (and elliptical) cylinder

\( I_n \) - Modified Bessel function of first kind of order \( n \)

\( I_n' \) - Derivative of the modified Bessel function of first kind of order \( n \)

\( J_n \) - \( n^{th} \) Bessel function of first kind

\( J_n' \) - Derivative of the \( n^{th} \) Bessel function of first kind

\( k_{nm} \) - \( m^{th} \) root of \( J_n'(k_{nm}a) \)

\( u, v, z \) - Position (in elliptic coordinates) at which a solution is sought

\( u_0, v_0, z_0 \) - Current source location in elliptic coordinates
\( e \) - eccentricity

\( J_{e_n} \) - \( n \)th radial first kind even Mathieu’s function

\( J_{o_n} \) - \( n \)th radial first kind odd Mathieu’s function

\( S_{e_n} \) - \( n \)th circumferential first kind even Mathieu’s function

\( S_{o_n} \) - \( n \)th circumferential first kind odd Mathieu’s function

\( h_{e_{nm}} \) - \( m \)th root of \( J'_{e_n}(h, \cosh(u)) = 0 \)

\( h_{o_{nm}} \) - \( m \)th root of \( J'_{o_n}(h, \cosh(u)) = 0 \)
Abstract

Electrical Impedance Tomography (EIT) is an emerging imaging technique with applications in the medical field and in the field of industrial process tomography (IPT). Until recently, data acquisition and image reconstruction schemes have been constructed with the assumption that the object being imaged is two-dimensional.

In recent years, some research groups have started to address the third dimensional aspects of EIT by both building three dimensional enabled data acquisition systems and solving the three dimensional Forward Problem numerically since this allows the possibility of modelling complex shapes. However, solving the Forward Problem analytically is still very attractive as an analytical solution does not depend on the way the domain has been meshed. Furthermore, if dynamic images are reconstructed which are less sensitive to the model of the electrodes employed, the shape of the object being imaged and the position of the electrodes, an analytical solution to the Forward Problem can be used to reconstruct dynamic three dimensional images. This thesis will start by describing how a full analytical solution for a finite right circular cylinder (which approximately models the human thorax) on which two electrodes have been placed, is derived. It will be shown that the analytical solution has two different forms. Results will be presented detailing the convergence performance of the two different forms as well as comparisons between the analytical solution and experimentally obtained data. Finally three dimensional images reconstructed using these methods will be presented.

In order to better approximate the shape of the human thorax, the above work has been extended to provide an analytical solution for an elliptical cylinder and this is presented in this thesis for the first time together with some simulation results.

Today in Multi-frequency Electrical Impedance Tomography (MEIT), new hardware for recording measurements operating above 1 MHz is now available. This high operating frequency raises the question of the validity of the employed quasi-static conditions used in the associated Forward Problem modelling. It is important to be able to determine when the quasi-static conditions fail and to investigate the differences between a solution
to the Forward Problem based on quasi-static conditions and the one based on non quasi-
static conditions at these frequencies. This thesis details the derivation of a new ana-
lytical solution based on non quasi-static conditions for a finite right circular cylinder
having two electrodes placed on its boundary. Some comparisons between the new an-
alytical solution and data obtained from in-vitro experiments will be presented in this
thesis. A comparison between the new analytical solution and the analytical solution de-
duced earlier in this thesis (which is based on quasi-static conditions) is also conducted.
Whilst these results are preliminary results, they reveal that for situations associated with
imaging the human thorax the quasi-static assumption appear violated when most mod-
ern MEIT systems are employed. This frequency dependent three dimensional analytical
Forward Problem work has wide ranging implications for the future of MEIT.
The thesis will conclude with some initial thoughts on how to incorporate anisotropy into
three dimensional Forward Problem solutions.
Chapter 1

Introduction

1.1 Medical Imaging Modalities

At the beginning of the 1980s, four tomographic medical imaging modalities were used for diagnosing diseases. These were

(1) X-ray Computerised Tomography (X-ray CT),

(2) Positron Emission Tomography (PET) or Single Positron Emission Computed Tomography (SPECT),

(3) Magnetic Resonance Imaging (MRI),

(4) Ultrasound.

The first technique known as X-ray Computerised Tomography (X-ray CT), in which the resulting image looks as though a planar slice (or tomograph) of the body had been physically removed and then radiographed by passing X-rays through it in a direction perpendicular to its plane, had been developed in the early 1970s. Such an image shows the human anatomy with a spatial resolution of about 1mm and a density discrimination of about 1%. This technique is also called transmission computed tomography (TCT) because the image is obtained by detecting the attenuated X-rays transmitted through the body. This technique provides information about anatomical details of the body organs (NRC-IOM, 1996).

A second group of techniques is called emission computed tomography (ECT) (NRC-IOM, 1996) which includes PET and SPECT. These techniques rely on the administration, either by injection or by inhalation, of radionuclide-labelled agents known as radiopharmaceuticals. Their distribution in the body of the patient depends on factors
such as blood flow, metabolic processes, etc. Then a map of this distribution is obtained by detecting the $\gamma$-rays produced by the decay of radionuclides. Therefore, ECT techniques yield functional information, in the sense that the images produced by ECT show the function of the biological tissues of the organs. Two different modalities of ECT exist:

(a) single photon emission computed tomography (SPECT), which makes use of radioisotopes such as $^{99m}$Tc, where a single $\gamma$-ray is emitted per nuclear disintegration.
(b) positron emission tomography (PET) which makes use of $\beta^+$-emitters like $^{11}$C, where the final result of a nuclear disintegration is a pair of $\gamma$-rays, propagating in opposite directions, produced by the annihilation of the positron in the tissue.

The third technique is Magnetic Resonance Imaging (MRI). MRI is not a transmission technique since the material itself is the signal source, i.e. the macroscopic spin magnetization $M$ from polarized water protons or other nuclei such as $^{23}$Na or $^{31}$P. The nuclear moments are randomly oriented, but they align when placed in a strong magnetic field. The principle of this technique is to measure the moment while it oscillates in a plane perpendicular to the static field (Conolly et al., 1995). By noticing that the moment experiences a torque proportional to the strength of the static magnetic field and it always points perpendicular to the static field, the spin then oscillates or precesses in a plane perpendicular to the static field. Since the precessing moments constitute a time-varying flux, they produce a measurable voltage in a loop antenna arranged to receive the $x$ and $y$ components of induction. Therefore, with MRI, it is possible to measure induction from the precessing nuclear moments of water protons. As the Nuclear Magnetic Resonance (NMR) signal from a human is due predominantly to water protons, and since these protons exist in an identical magnetic environment, they all resonate at the same frequency. Therefore, the NMR signal is simply proportional to the volume of water. To obtain images, MRI relies on the imposition of spatial variations in the magnetic field to distinguish spins by their location. Applying a magnetic field gradient causes each region of the volume to oscillate at a distinct frequency. Fourier analysis of the signals obtains a
map of the spatial distribution of spins. By the use of the k-space, it is then possible to reconstruct images with high spatial resolution.

The fourth technique is Ultrasound. In this technique an ultrasonic pressure pulse, approximately a microsecond long, is directed into the tissue by a transducer consisting of an array of individually pulsed piezoelectric elements. This pulse is reflected from various scatters and reflectors within the tissue under investigation. The scattered pressure wave is detected by the transducer array and focused using electronic beam forming. The resulting signals are used to make an image that correlates to the scatters and reflectors within the region from which the pressure pulse signal was reflected. Ultrasound is currently only a two-dimensional imaging modality, although many investigators are researching three-dimensional imaging or are combining flow with scattering images in several dimensions (Bamber, 1999). Ultrasound is a real-time tomographic imaging modality. Not only does it produce real-time tomograms of scattering, but it can also be used to produce real-time images of tissue and blood motion, elasticity, and flow in the tissue (NRC-IOM, 1996).

All these four medical imaging systems rely on the same concept of imaging using the attenuation data resulting from the interaction between the biological tissue and the energy applied to it in the form of a field or a wave.

During the 1970s, a few researchers proposed the use of electrical current for producing images of biological tissues. This was indirectly an extension of the work done on the modelling of biological tissues by analyzing their electrical properties. Therefore, a new medical imaging modality was born. This medical imaging modality took different names during the end of the 1970s and the beginning of the 1980s. Today, this imaging modality is known as Electrical Impedance Tomography (EIT). Its origins can be traced back to Price’s paper (Price, 1979) who proposed to use a new CT imaging technique based on the principle of using guard electrodes to make beams of current rather than beams of X-rays. The use of guard electrodes was an attempt to make the beam of current as parallel as possible. The main motivation for this concept appeared to be the reuse of X-ray CT image reconstruction algorithms. The analogy between the X-ray CT
reconstruction algorithms and EIT algorithms could be seen very clearly at the beginning
of the 1980s and the first group to produce in-vivo tomographic impedance images was
the Sheffield group. The idea behind their image reconstruction algorithm was based
on the back-projection concept used in X-ray CT technique (Boone et al, 1997). Their
technique will be described very briefly in the next section.

Table 1.1 summarizes the different medical imaging systems according to their advan-
tages and disadvantages.

<table>
<thead>
<tr>
<th>Techniques</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>X-rays CT</td>
<td>Good spatial resolution, Widely used in medical applications, Not too expensive, 3D images from 2D slices</td>
<td>Hazard (ionising radiation), Not portable, No real-time monitoring</td>
</tr>
<tr>
<td>PET SPECT</td>
<td>Functional imaging, 3D images from 2D projections, Widely used in medical applications</td>
<td>Hazard, Low spatial resolution, Not portable, No real-time monitoring, Very expensive</td>
</tr>
<tr>
<td>MRI</td>
<td>High spatial resolution, Widely used in medical applications, Good soft tissue discrimination</td>
<td>Not portable, No real-time monitoring, Expensive</td>
</tr>
<tr>
<td>Ultrasound</td>
<td>Portable, Real Time monitoring, Cheap</td>
<td>Low spatial resolution, Not easy to reconstruct 3D images</td>
</tr>
<tr>
<td>EIT</td>
<td>Portable, Real Time monitoring, Cheap</td>
<td>Low spatial resolution, Not easy to reconstruct 3D images</td>
</tr>
</tbody>
</table>

Table 1.1: Advantages and disadvantages of different medical imaging modalities
Today, EIT is becoming a promising technique which offers the possibility of determining time-varying electrical properties (conductivity and permittivity) within a body from applied electrical currents and observed voltages on the surface of the body. Differences in the electrical properties of components within the body can be used to identify objects and reconstruct internal images. The uses of EIT as a medical application range from locating pulmonary embolisms to monitoring heart function (Metherall, 1998), (Boone et al, 1997). Perhaps, in the future, EIT could provide information and images by far less intrusive techniques than those described above. However, one of the major disadvantages of EIT compared with the other techniques is that the problem to be solved mathematically in order to reconstruct an image is a non-linear ill-posed inverse problem.

The remaining sections in this introductory chapter will review briefly EIT by explaining the principle of EIT, the data collection methods, the image reconstruction algorithms and Multi-frequency EIT. This chapter ends with the aims and objectives of this thesis and a framework for the work reported.

1.2 The Principle of Image Reconstruction in EIT

1.2.1 Inverse Problems

The principle of EIT can be understood by applying two current carrying electrodes on one part of the human body and trying to recover the resistance of that part of the human body from information contained in measurements taken by recording the potential difference between two other electrodes (see figure 1.1).
In figure 1.1, the relationship between the injected current \( (I) \), the resistivity \( (R) \) and the voltage measurement \( (V) \) is described by Ohm’s law (Marshall and Skitek, 1990),

\[
V = RI
\]  

(1.1)

where
\[
V : \text{voltage measurement}, \quad I : \text{injected current}, \quad R : \text{resistance}.
\]

By injecting a current \( I \) on two electrodes and recording the voltage \( V \) on the two other electrodes, the resistance can be found. Having found the resistance, if the shape of the object is known, the conductivity can be determined subject to certain constrains(such as the medium being uniform and isotropic). In this example, the two main steps in the image reconstruction in EIT can be seen. The first step is known as the *Forward Problem* whereby knowing the conductivity distribution, the topology of the object being imaged and the injected current, the boundary voltage profile can be determined (see figure 1.2). In order to produce an image of the conductivity distribution, the conductivity distribution must be discretized into pixels which will then represent the conductivity distribution in the image space.
However, one measurement is usually insufficient, since each pixel in the image space may have a value different to the value of its neighbour. Therefore, a number of electrodes are placed around the object being imaged in order to produce an independent set of boundary voltage measurements. In other words, the second step in the process of generating an image consists of determining the conductivity distributions and discretizing it into pixels by knowing the topology, the injected current and the set of boundary voltage profiles recorded around the surface of the object being imaged. This second step is called the Inverse Problem (see figure 1.3). It will be seen later in this introductory chapter that there are a number of different algorithms used in EIT for reconstructing
an image. Furthermore, the method of injecting the current and recording the voltages which give rise to a number of independent measurement, can also vary. A number of data collection methods (DCM) have been proposed during the last twenty years and the next section will summarize some of them.

Because of the electrode effects (and the electrical safety) associated with the use of direct current (DC), all the equipment in EIT uses an alternating current (AC) mostly in the frequency range of approximately 1 kHz to 1 MHz. The magnitude of the impedance is found by the voltmeter and the resistive and reactive components are found by the use of a synchronous demodulator (Boone et al, 1997).

Furthermore, the relationship between the voltage measurements and the current injected is not straightforward since it is subject to non-linear relationship governing the voltage and the current flow within the object (Barber, 1995). This makes the Inverse Problem non-linear. Image reconstruction is challenging since large changes in the conductivity distribution may only lead to very small changes in the boundary measurements. The problem is therefore said to be both non-linear and ill-posed and as a consequence, regularization methods such as the standard Tikhonov regularization, or truncated Singular Value Decomposition have to be used in the image reconstruction algorithms.

1.3 Data Collection Methods

So far little has been said on how to inject the current and to record the voltage. The simplest current patterns to use are those given by passing current into the object through one electrode and extracting current through a second electrode (bipolar pattern). The bipolar pattern will be reviewed later in this section when the different tetrapolar strategies are reviewed. However, other current application patterns are possible. Current can be passed simultaneously through many electrode, with different amounts passing through each electrode. Since there are many different strategies, it is important to de-
fine an optimal strategy. In 1986, Isaacson defined the optimal density current strategy. This strategy will be firstly reviewed in this section. It will also be explained, later in this thesis, why this strategy is not used.

1.3.1 Optimal Density Current Strategy

The optimal density current strategy (Rigaud et al, 1996) is based on the concept that if two conductivity distributions have to be distinguished, then there exists an optimal current density pattern which maximizes the signal-to-noise ratio in the reconstructed images and therefore maximizes the distinguishability between these two conductivity distributions. This principle can be understood by realizing that the optimal current density induces in the imaged zone a current density distribution which is more uniform than the one induced by other current patterns and, therefore, improves the sensitivity and the quality of data obtained. The optimal patterns are often cosine-like patterns of current amplitude distributed around the object boundary rather than being localized at a pair of points, as in the two-electrode case. Since the currents are passed simultaneously through many electrodes, it is tempting to try and use the same electrodes for voltage measurements (this technique is also known as Adaptive Current Tomography (Gisser et al, 1988)). This produces two problems. Firstly, there is still a question mark associated with this method, since the voltage measurements will be sensitive to the unknown contact impedance which is formed at the electrode/skin interface. This will cause a significant unknown voltage drop across the interface and errors will therefore be introduced as the measured voltages will not equal the surface voltages which are required for image reconstruction. Although Isaacson has argued that electrode impedance can be included in the reconstruction algorithm, there is little evidence that satisfactory in-vivo results can be obtained using data from simultaneous voltage/current measurements. Secondly, it has proved difficult to model current flow around an electrode through which current is flowing with sufficient accuracy to allow the reliable calculation of voltage on that electrode, as needed for accurate reconstruction. It seems that separate electrodes should be used for voltage measurements with distributed current system. Because of
these two reasons, the optimal density current strategy will not be pursued further in the thesis.

1.3.2 Tetrapolar Strategy

The tetrapolar strategy is really based on the bipolar patterns where one electrode injects a current and a second electrode is used for sinking the current. Along with these two injecting electrodes, a number of other electrodes are placed around the object being imaged which only record the voltage. The pattern for injecting the current and recording the voltage is summarized by the tetrapolar strategy in which different configurations exist such as:

(1) Adjacent Configuration
(2) Polar Configuration
(3) Interleaved Configuration

All these configurations have two common properties. Namely, the electrodes are equally spaced around the object being imaged and the electrode used for recording the voltage is not the same as the one for injecting the current (four electrodes model). Figure 1.4 shows the different configurations in two dimensions when only eight electrodes are used. Each of these configurations has its advantages and disadvantages.
Further details of these strategies can be found in (Rigaud et al, 1996). As the tetrapolar strategy is more simple to model than the optimal density current strategy (allowing simple electrode models to be used) and as the injecting electrodes are not used for recording the voltages, this strategy will be used in the thesis.

1.3.3 Spatial Resolution of images in EIT and number of independent measurements

The spatial resolution of the image is dependent on a number of aspects, namely the number of independent measurements and the noise generated by the instruments and the algorithms. The number of independent measurements is very important for the resolution of image since the more independent boundary measurements that are collected, the greater will be the number of independent conductivity elements that can be theoretically resolved. Hence the resolution will be improved. Since the tetrapolar strategy is used in the thesis, an example is described, in the next paragraph, of how the number of independent measurements are computed for the adjacent configuration.

In this collection method, N equally spaced electrodes are placed around the boundary of an object in one plane and N-1 independent measurements can be made. Since the measurements of voltage concerning the drive electrode have to be ignored only $N - 3$
independent measurements are made. To have the complete set of independent measurements, current is passed through all of the adjacent electrode pairs in turn. This would give $N(N - 3)$ measurements. However, the principle of reciprocity\(^1\) will reduce this number of independent measurements to $\frac{N(N-3)}{2}$.

### 1.4 Type of Images in EIT

As described earlier in this chapter, the complete reconstruction problem in EIT is a non-linear ill-posed problem (Vauhkonen, 1997). This makes the image reconstruction algorithm challenging and this also explains why there are so many different algorithms around today. However, it is possible to classify these algorithms according to the type of image they produce, and also, according to the type of image reconstruction algorithm they employ. Figure 1.5 shows the different type of images that exist in EIT and the type of algorithms which are used for producing the images. The image reconstruction process in EIT can be performed either in a single step or iteratively. Reconstructing the full non-linear problem in one single step is very challenging and few algorithms attempt this, an exception being the layer stripping algorithm proposed by (Cheney et al, 1991). Another way of solving the full non-linear problem is to employ an iterative algorithm where each step in the iterative process can be processed linearly (Barber, 1992). In the following sections, only these image reconstruction algorithms will be reviewed.

---

\(^1\) Principle of reciprocities states that reversing the voltage measurement and current injecting electrodes would give an identical value of resistivity (Webster, 1990)
Iterative algorithms aim to reconstruct static images, i.e. find the actual conductivity in the body rather than a change in conductivity. The most popular iterative algorithms are variants of the Newton-Raphson method (Woo, 1990), (Cheney et al, 1990). There are two major limitations to iterative algorithms. Firstly, the reconstruction algorithm must include an accurate model of the physical system to be imaged. It must also include the exact positions and shapes of the skin electrodes, and the electrode-skin impedance. Basically, the model used in iterative algorithms must be very accurate in order to allow the iterative solutions to converge on a correct solution. So far, this accuracy has not been achieved. In 1990, Cheney et al proposed to reconstruct an image using only the first step in the iterative algorithm. This algorithm is known as NOSER algorithm.
(Cheney et al., 1990). Some images have been generated using this algorithm. The second major limitation is that the iterative process is particularly sensitive to noise and measurement error. These two major limitations make the reconstruction of static images very difficult (Boone et al., 1997).

Because of these two major limitations, iterative algorithms at present are difficult for performing in-vivo studies. This restricts very much the use of iterative algorithms for medical applications. This explains partially why noniterative algorithms have been very popular in the medical field since they are good at reconstructing in-vivo measurements, albeit with some constraints. This is mainly due to the fact that noniterative algorithms are less restrictive than iterative algorithms in terms of model (i.e., the electrode model does not have to be as accurate as in iterative algorithms) and in terms of position of electrodes. It is also the reason why the work presented in the thesis will focus mainly on noniterative algorithms. However, some iterative algorithms will be reviewed in chapter 2.

1.4.2 Noniterative algorithms

Noniterative algorithms are based on the fact that each step in an iterative algorithm is linear. Indeed, images reconstructed using the first step of the iteration process treat image formation as a linear process (which is an assumption justified for small changes in conductivity from the uniform conductivity). It is also the reason why they are often called single step algorithms. In noniterative algorithms like the NOSER approach, only one single step is used for generating an image. Noniterative algorithms are based on the principle that if a change in the conductivity occurs within the object, then it can often be assumed that the change in surface voltage is dominated by this conductivity change. As a consequence, noniterative algorithms produce only dynamic images (also called differential images) with the aim to image changes in conductivity rather than absolute values. In many cases, two datasets are recorded at two different times, and one dataset is used for normalising the other one. This method is explained in the following example:
Suppose that the voltage difference between a pair of electrodes before a conductivity change occurs is $g_1$ and that the value after a conductivity change occurs is $g_2$, then a normalized datum value is defined as

$$
\Delta g_n = \frac{g_1 - g_2}{g_1} = \frac{\Delta g}{g_1}
$$

(1.2)

This normalisation process explains why, to a large extent, noniterative algorithms are often very robust since the conductivity is normalized with the advantage of reducing the effects of body shape and electrode configuration as they cancel out. Nevertheless, they have the disadvantage of imaging only conductivity changes and therefore they do not provide any quantitative results.

Chapter 2 will review noniterative algorithms with special attention paid to the Sheffield’s filtered back-projection algorithm.

### 1.4.3 Multi-frequency EIT (MEIT)

As described above, noniterative algorithms can only image changes in conductivity. Absolute distributions of conductivity cannot be produced using these methods. In addition, any gross movement of the electrodes, either because they have to be removed and replaced or even because of significant patient movement, makes the use of this technique difficult for long-term measurement of changes. However, instead of recording two sets of measurements in the time domain, two sets of measurements could be recorded in the frequency domain, thereby producing dynamic images in the frequency domain. Assuming the datasets acquired for producing the dynamic images in the frequency domain were collected within a short time interval in the time domain, the resulting images could be classified as quasi-static images.

Another kind of image can be produced in EIT by looking at the behaviour of the current in a biological tissue across a range of frequencies. A common feature of biological tissue is that it consists of different types of cells which are connected together. The cell’s interior containing various structures in a conducting fluid (called cytosol) is enclosed
by an electrically insulating membrane. The extracellular space also contains an electrolyte normally exhibiting a lower resistivity than the cytosol. The membrane is formed by a lipid bilayer (3-5 nm thickness) which is composed of molecules exhibiting both hydrophilic and hydrophobic properties. In the presence of water, these molecules can form a bimolecular leaflet, whose polar heads are facing outward and inward such that hydrophobic parts make up the inside of the membrane. This structure is not rigid, allowing the macromolecules distributed over the membrane surface to be inserted in the lipid bilayer. This makes the structure to be composed of different permittivities. Therefore, the cell membrane behaves as a dielectric interface and can be assimilated with the two armatures of a capacitor. This explains why, at low frequencies of applied current, the current cannot pass through the membranes, and conduction is through the extracellular space. However, current can flow through the membranes (acting as a capacitor) at higher frequencies (see figure 1.6).

![Cole-Cole circuit model](image)

Figure 1.6. The Cole-Cole representation of bulk tissue impedance

Based on this, Cole-Cole (Cole and Cole, 1941) derived a simple electrical model which is a combination of two resistances and one capacitance that models the above behaviour
of tissue. Clearly, this model as it stands is too simple, since an actual tissue sample would be better represented as a large network of interconnected modules of this form. However, it has been shown that this model fits experimental data if the values of the components are made a power function of the applied frequency $\omega$. The components of the Cole-Cole model can be found by making measurements of the real and imaginary components of tissue impedance over a range of frequencies. Furthermore, it also known that tissue structure alters in disease. Since $R$, $S$, and $C$ are dependent on structure, it should be possible to use such measurements to distinguish different types of tissue and different disease conditions. As a consequence, different research groups have shown that if measurements are made over a range of frequencies and differential images produced using data from the lowest frequency, and the other frequencies in turn, these images can be used to compute *parametric* images representing the distribution of the combination of the Cole-Cole circuit values. The technique for computing the parametric images is based on the Cole-Cole equation and will be briefly introduced in the next chapter.

Parametric images are images of absolute tissue properties. Since these properties are related to tissue structure, they should produce images with useful contrast. Furthermore, it is possible to collect multifrequency data and to reconstruct an image in a short time to preclude significant patient movement, which means these images should be robust against movement artifacts. Change of these parameters with time can still be observed.

### 1.5 Purposes and structure of the thesis

At the beginning of the 1990s, there was a major problem with image reconstruction algorithms since they reconstructed only two dimensional objects. This was mainly due to limitations with the hardware. However, it was also clear that most *in-vivo* studies such as imaging the human thorax violate this assumption. In severe cases, the complex three-dimensional structure and resulting current flow lead to gross image distortions.

---

2 Impedivity: The specific impedance of an electrically conducting material. The inverse of admittivity.
when images are reconstructed using existing two dimensional algorithm. The reason is that there are significant contributions to the image from off-plane conductivity changes. Therefore, in reality, the current is not constrained to only the plane where the electrodes are positioned, but also propagates throughout the volume surrounding the plane of electrodes. Since no effective means exist to collimate the data arising from a three-dimensional object as some other imaging modality and hence provide a two dimensional image uncontaminated from off-plane structures, there appeared to be little option but to address the three dimensional aspect of EIT. The construction of a three dimensional image reconstruction algorithm involves the following stages:

1) solution to the Forward Problem,
2) solution of the Inverse Problem,
3) displaying the image in three dimensions.

Based on that, the first aim of this thesis is to model the Forward Problem in three dimensions. Most of the research groups which have solved the Forward Problem in three dimensions, solved it numerically. However, in this thesis, the Forward Problem will be solved analytically. The reason for solving it analytically will be explained in chapter 3. The second aim of this thesis will demonstrate how the analytical tools used for finding an analytical solution to the Forward Problem in three dimensions using a simple geometrical shape can also be used for finding an analytical solution to a Forward Problem which models a more complex geometrical shape. The third aim of this thesis is concerned with Multi-Frequency EIT (MEIT). MEIT works under quasi-static conditions, meaning that the range of frequencies used starts from a very low frequency (10 Hz) to about 1 MHz. Nevertheless, a few research groups have started to use frequencies above 1 MHz which raises the question about the validity of the quasi-static conditions employed. Therefore, having already modelled the Forward Problem in three dimensions, the fourth aim of the thesis is to extend the solution to the Forward Problem to multi-frequency in an attempt to explore the limits of the quasi-static assumptions of MEIT.
1.5.1 Purposes of the thesis

Based on the above, the objectives of the work reported in this thesis are:

1) To address the three dimensional aspect of EIT by modelling analytically the Forward Problem,

2) To reconstruct images of a three dimensional object,

3) To use the analytical tools used in (1) for finding an analytical solution to the Forward Problem which models a more complex geometrical shape,

4) To present some new methods for improving the analytical solution to the Forward Problem, i.e:
   a. To extend the work done in quasi-static conditions to Multi-frequency EIT,
   b. To describe how anisotropy can be taken into account.

1.5.2 Structure of the thesis

The thesis is divided into four sections.

The first section is concerned with introducing the governing equation for EIT and the different image reconstruction algorithms for EIT and MEIT. Then, the tools used for solving the Forward Problem will be presented. This section contains:

Chapter 2 introduces the governing equations for EIT, together with the assumptions which allow them to be simplified, followed by a brief description of the different image reconstruction algorithms for EIT and MEIT.

Chapter 3 starts by discussing how the Forward Problem will be solved i.e., analytically or numerically. Then, the different tools for solving the Forward Problem are presented.

The second section addresses the first aim of the thesis by solving the Forward Problem in three dimensions, showing some image reconstructions and finally presenting some comparisons with some experimental values. This section contains:
Chapter 4 shows the derivation of the analytical solution to the Forward Problem for a finite right circular cylinder. Then, a study of convergence and reconstruction of the equipotentials are presented. This chapter finishes by presenting a validation of the analytical solution against other Forward Problem solvers.

Chapter 5 presents the algorithm used for reconstructing an image. Then, it shows some images reconstructed by the use of the analytical solution for a finite right circular cylinder.

The third section describes how the simple geometrical model (a finite right circular cylinder) used for solving analytically the Forward Problem (see chapter 4) can be extended into a more complex model (elliptical cylinder).

Chapter 6 presents the derivation of the analytical solution for an elliptical cylinder and presents some results.

The main part of the thesis has been presented from chapter 3 to chapter 6. These chapters are concerned with EIT when the quasi-static conditions (see chapter 2) have been applied.

The fourth section addresses the issue when the quasi-static assumptions break and will propose an alternative way of modelling the Forward Problem.

Chapter 7 determines when the quasi-static conditions break down and how to model the Forward Problem for a finite right circular cylinder using the analytical tools presented in the previous chapters when the quasi-static assumptions are lifted.

Chapter 8: Future Work and Conclusion starts by describing some future work to extend the work described in the thesis. For instance how anisotropy could be modelled with the same technique used for modelling the Forward Problem under isotropic conditions. This chapter will finish by summarizing what has been achieved and what are the remaining limitations.

Finally, the work presented in this thesis has been reported in the following publications:


(3) Kleinermann F, Avis N J Forward and Inverse Problem Solutions for three dimensional Electrical Impedance Tomography, IEE, 5.1-5.3, 1996

(4) Kleinermann F, Avis N J A comparison of voltage and sensitivity profiles from two forward solvers for three dimensional electrical impedance tomography, Conference proceeding for the 18th annual conference of the IEEE Engineering in Medicine and Biology Society, 743, 1996


(6) Kleinermann F and Avis N J, Frequency dependent forward problem solvers for multifrequency electrical impedance tomography, Conference proceedings for the World Congress on Medical Physics and Biomedical Engineering, 323, 1997


(8) Kleinermann F, Avis N J and Alharghan F A, Analytical solution to the three-dimensional electric forward problem for a circular cylinder, Conference proceedings of 1st World Congress on Industrial Process Tomography, Buxton, April, 1999

(9) Kleinermann F, Avis N J and Alharghan F A, Analytical solution to the three-dimensional electric forward problem for a circular cylinder, abstract for the 1st EPSRC Engineering Network meeting on Biomedical applications of EIT, April 14-16, 1999

Chapter 2

Image Reconstruction Algorithms for EIT

As described in chapter 1, the aim of Electrical Impedance Tomography is to determine the internal conductivity distribution of a bounded region from measurements made at the surface. These measurements are usually the surface voltages generated as a result of injecting known current patterns into the region, although some research groups (Kim and Woo, 1987) have preferred to measure the boundary currents from applied voltage patterns.

Furthermore, Kohn and Vogelius (Kohn and Vogelius, 1984, 1985) and Sylvester and Uhlmann (Sylvester and Uhlmann, 1986) showed that for domains of dimensionality greater than one, boundary measurements can uniquely determine the internal isotropic conductivity distribution, so long as the measurements are made to infinite precision and that they encompass the entire surface. In practice these conditions are not met due to the limitations in measurement accuracy and the finite number of measurements that can be made at discrete locations on the surface, namely the electrode sites.

It must also be pointed out that many biological tissues exhibit anisotropic behaviour in which case the unique solution may fail as two anisotropic conductivity distributions can lead to the same boundary measurements (Lionheart, 1995). Although this clearly presents a problem in EIT, the relatively small sample of boundary measurements and their limited accuracy pose the most significant practical issues (Barber, 1992). Modelling anisotropy will be briefly addressed in the Future Work section at the end of the thesis where details of how to model the Forward Problem with anisotropic materials will be briefly discussed.
In this chapter, the main algorithms used for iterative and noniterative image reconstructions are reviewed. It must also be said that most of these algorithms have been applied only to the two dimensional image reconstruction problem, although in a small number of studies these algorithms have been adapted for the three dimensional cases.

### 2.1 Governing equations for EIT

In order to reconstruct an image from a set of boundary voltage measurements, a physical model which connects the measured voltages, injected currents and a conductivity distribution needs to be derived and this is done by the use of equations associated with some boundary conditions. The underlying relationships which govern the interaction of electricity and magnetism are summarized by Maxwell’s equations. However, several simplifications can be applied in EIT which simplify the complexity of the problem. Maxwell’s equations for the time domain in inhomogeneous media when there are no magnetic sources (Marshall and Skitek, 1990) are:

\[
\begin{align*}
\nabla \times H & = J + \frac{\partial D}{\partial t} \\
\nabla \times E & = -\frac{\partial B}{\partial t} \\
\nabla \cdot J & = -\frac{\partial \rho}{\partial t} \\
\nabla \cdot B & = 0 \\
\nabla \cdot D & = \rho
\end{align*}
\]

where

- $H$ : Magnetic field,
- $E$ : Electric field,
- $B$ : Magnetic induction,
- $D$ : Electric displacement,
- $J$ : Electric current density,
- $\rho$ : Charge density.
In EIT, the tissues are usually approximated as isotropic even though some biological tissues such as the muscle are highly anisotropic (Geddes and Baker, 1967). Anisotropic considerations can be found in (Lee and Uhlmann, 1989), (Sylvester, 1990) (Breckon, 1992), (Glidewell and Ng, 1995).

If the injected currents are time-harmonic with frequency $\omega$ the electric and the magnetic fields are of the form

$$E = \hat{E} e^{i\omega t} \text{ and } H = \hat{H} e^{i\omega t}$$  \hspace{1cm} (2.6)

Furthermore, in a linear medium, the following relationships are valid

$$D = \epsilon E$$  \hspace{1cm} (2.7)

$$B = \mu H$$  \hspace{1cm} (2.8)

Biological tissues are mostly nonmagnetic and thus the permeability is only the free space permeability ($\mu_0 = 1.2566370614 \times 10^{-6} \text{ N A}^{-2}$). The magnetic flux density $B$ (2.8) becomes:

$$B = \mu_0 H$$  \hspace{1cm} (2.9)

In EIT, as in any other bioelectric phenomena (Malmivuo and Plonsey, 1995), there are current sources. Therefore, the total current density is given by $J = J_0 + J_{\text{impressed}}$ where $J_0 = \sigma E$ is the so-called ohmic current and where $J_{\text{impressed}}$ are the current sources. Using the relations (2.7-2.9) and assuming that the injected currents are time-harmonic, the equations (2.1-2.2) can be written in the form:

$$\nabla \times E = -i\omega \mu_0 H$$  \hspace{1cm} (2.10)

$$\nabla \times H - (i\omega \epsilon(\omega) + \sigma(\omega)) E = J_{\text{impressed}}$$  \hspace{1cm} (2.11)

where $\sigma(\omega) + i\omega \epsilon(\omega)$ is called the admittivity.
Equation (2.10) and equation (2.11) are the full Maxwell’s equations. Furthermore, the electric field can be found as

\[ E = -\nabla \Phi - \frac{\partial A}{\partial t} \]  \hspace{1cm} \text{in the time domain} \tag{2.12}

\[ E' = -\nabla \Phi - i\omega A \]  \hspace{1cm} \text{in the frequency domain} \tag{2.13}

where \( A \) is the magnetic vector potential and \( \Phi \) is the electric potential.

The exact derivation of \( E \) is described in more details in the chapter seven when multi-frequency EIT is addressed. The reader is referred to that chapter for the derivation of \( E \).

In EIT some simplifications to the full Maxwell’s equations are made. The first assumption is that the second term in (2.12) or (2.13) involving the time or the frequency is neglected. This means that the effect of magnetic induction that causes the induced electric field is neglected. In order to know when this assumption is valid, a condition must be defined. This condition (Nunez, 1981) is defined as

\[ \omega \mu \sigma L_c^2 (1 + \frac{\omega \varepsilon}{\sigma}) \ll 1 \tag{2.14} \]

where \( L_c \) is a characteristic distance over which \( E \) varies significantly, and the effect of magnetic induction can be neglected.

The second assumption often associated with EIT is to neglect the capacitive effects expressed by the term \( i\omega E \) in equation (2.11). The condition for which this second assumption is valid (Nunez, 1981) is defined as

\[ \frac{\omega \varepsilon}{\sigma} \ll 1 \tag{2.15} \]

These two assumptions are called quasi-static assumptions. An example of how the quasi-static conditions (equation (2.14) and equation (2.15)) are justified, is described by the following situation. Suppose that the quasi-static conditions are applied to the human thorax having a diameter of 0.4m (\( L_c \)) and mainly composed of skeletal muscle. Furthermore, the human thorax is assumed to be isotropic. For such a situation, the
magnetic permeability is $\mu \simeq \mu_0$, the conductivity is $\sigma \simeq 4 \, \text{Sm}^{-1}$, the permittivity is $\varepsilon \simeq 8.85 \times 10^{-7} \, \frac{\text{C}^2}{\text{Nm}^2}$, and the frequency is $f \simeq 10$ kHz. With these values the condition for neglecting the magnetic effects become:

$$\omega \mu \sigma L_c^2 (1 + \frac{\omega \varepsilon}{\sigma}) \approx 0.004 \ll 1$$  \hspace{1cm} (2.16)

Therefore neglecting of magnetic induction can be considered as a good assumption (at least for the situation outlined above).

Applying the above parameters to the second quasi-static condition (equation 2.15), the condition of neglecting the capacitive effects becomes

$$\frac{\omega \varepsilon}{\sigma} \approx 0.22$$  \hspace{1cm} (2.17)

Therefore, according to the condition (2.15), the second assumption is also valid. Whilst this strictly satisfies the condition (2.15), the situation is not as clear cut as above since 0.22 is not much less than 1 (Vauhkonen, 1997). In chapter 7, the situation will be investigated further for different tissues.

In the quasi-static case, the modified Maxwell’s equations in linear, isotropic medium are

$$E = -\nabla \Phi$$  \hspace{1cm} (2.18)

$$\nabla \times H = \sigma E + J_{\text{impressed}}$$  \hspace{1cm} (2.19)

Taking the divergence on both sides of equation (2.19) and substituting (2.18) into (2.19), the governing equation for EIT inside the body becomes

$$\nabla \cdot (\sigma \nabla \Phi) = \nabla \cdot J_{\text{impressed}}$$  \hspace{1cm} (2.20)
Furthermore, if there is no current inside the body and on the boundary of the body, equation (2.20) becomes

$$\nabla \cdot (\sigma \nabla \Phi) = 0$$  \hspace{1cm} (2.21)

Expanding the divergence of this product gives Poisson’s equation:

$$\sigma \nabla^2 \Phi + \nabla \sigma \cdot \nabla \Phi = 0$$ \hspace{1cm} (2.22)

The above equation is also Laplace’s equation: \(\nabla^2 \Phi = 0\).

As mentioned in chapter 1, the solution of equation (2.21) for a given conductivity distribution, is the Forward Problem. Furthermore, the distribution of \(\Phi\) is dependent on the conductivity distribution, the shape of the object and the position of the electrodes (Kiber et al, 1990).

The Forward Problem is an essential stage in the image reconstruction algorithms for both iterative and noniterative algorithms. In the next section, the different image reconstruction algorithms are reviewed.

### 2.2 Iterative algorithms

Iterative algorithms seek a full reconstruction of the conductivity distribution, using iterative methods to take account of the non-linearity of the inverse problem. They reconstruct static images of the absolute value of the conductivity with the advantage of good precision. The disadvantages of that approach are the time for reconstructing the conductivity distribution, the convergence of iterative process and, finally, the complexity associated with the image reconstruction algorithm. Figure 2.1 summarizes the different stages in most iterative image reconstruction algorithms.
Figure 2.1. Iterative algorithm

From figure 2.1, it can be seen that the Forward Problem is computed at each iteration which is why the reconstruction of images with iterative algorithms is slow. There are three main iterative algorithms reviewed in this section.

2.2.1 Perturbation algorithm

This technique is based on the application of a known voltage pattern and measurement of current (Kim et al., 1983), (Webster, 1990), (Yorkey et al., 1987). The idea is the creation of a perturbation matrix created by the repeated computation of the Forward Problem in order to correct an estimated conductivity distribution. A coefficient of the perturbation matrix $S_{ijl}$ was evaluated by computing the change in output current resulting from a ten times resistivity increase for a single element $i$ within the FEM mesh. The entire matrix was constructed by repeating this process for all elements and for each voltage projection $l$ and current measurement electrode $j$. Kim et al (1983) only stored the integer part of the sensitivities to limit the computational and the storage requirements of this technique, but also forced the large peripheral sensitivities to zero to overcome nonconvergence problems.
Assuming an initial uniform conductivity distribution, the iterative process calculates a new conductivity estimate by

$$\frac{1}{c_i^n} = \frac{1}{c_i^{n-1}} + f \sum_{j=1}^{m} \frac{c_{ij}S_{ij}}{|S_{ij}|} \left( \frac{1}{c_i^{n-1}} \right)$$

(2.23)

where

- $c_i^n$: conductivity of the $i^{th}$ element after $n^{th}$ and $(n - 1)^{th}$ iteration respectively,
- $f$: over-relaxation factor to speed up convergence,
- $c_{ij}$: percentage difference between the predicted (from the FEM) and actual (measured) current density at the $j^{th}$ electrode in the $n^{th}$ iteration,
- $m$: total number of current measurement electrodes.

The perturbation method can be seen as a kind of a recursive estimator since the conductivities are updated after each voltage pattern.

Kim et al (1983) presented images reconstructed from simulated data with a variety of starting conductivities estimates. Assuming an initial uniform conductivity distribution, reasonable images were obtained after 100 iterations. In Kim et al (1983), the perturbation matrix remains unchanged during the entire image reconstruction process. For this reason, Yorkey et al (1987) proposed a modification to this algorithm by recognizing that the perturbation matrix was a function of the conductivity distribution. They therefore recalculated the matrix at each iteration.

For Kim and Woo (1987) the perturbation matrix was also a function of element size, distance from the measuring electrodes and the background conductivity and therefore they developed a modified method whereby the conductivity updates were modulated based on a linearly decreasing function of the element to electrode distance. Despite all these modifications, only images reconstructed from simulated data, or with data collected from a saline filled phantom, have been demonstrated.
2.2.2 Double Constraint algorithm

Wexler et al (1985) developed an iterative technique which was later named the Double Constraint Method by Yorkey et al (1987). Starting with an assumed uniform conductivity, at each iteration a FEM Forward Problem solver is used successively by applying Neumann and then Cauchy boundary conditions.

In the first step of the algorithm, the internal and boundary voltages are calculated for the known current injection pattern and the latest conductivity estimate. The first step computes the electric current density in each element \( i \) by using

\[
J = c_i E
\]

(2.24)

In the second step, the potential distribution is again calculated with additional measured boundary voltages to constrain the solution. This gives a more accurate estimate of the internal voltage field \( \nabla \Phi \) and hence the interior potentials are nudged in the correct direction.

As the initial conductivity distribution \( c_i \) will not in general be correct, a discrepancy will exist between the voltage gradient calculated at the 1\textsuperscript{st} and 2\textsuperscript{nd} steps.

\[
J + c_i \nabla \Phi \neq 0
\]

(2.25)

The aim of this technique is therefore to iteratively adjust the conductivity distribution in order to minimize the squared residuals between the two solutions of the boundary value problems. That is to minimize:

\[
e(c) = \sum_i \int_{\Omega_i} [J + c_i \nabla \Phi] \cdot [J + c_i \nabla \Phi] \, dv
\]

(2.26)

where
$e(c)$ is the square of the residual sum of norms, 
$v_i$ and $dv$ denote the volume integral over the element $i$, 
$J$ is the electric current density in element $i$ found by solving
the Neumann boundary conditions,
$-c_i \nabla \Phi$ is the electric current density in element $i$ found by
solving the Cauchy boundary conditions.

By differentiating equation (2.26) with respect to conductivity and equating to zero, the
conductivity update for the $i^{th}$ element $c_i$ can be found from:

$$c_i = -\frac{\int_{v_i} J \cdot \nabla \Phi \, dv}{\int_{v_i} \nabla \Phi \cdot \nabla \Phi \, dv}$$  \hspace{1cm} (2.27)
Convergence is determined by comparing the calculated boundary voltages from the 1st step and the measured voltages, to a pre-determined threshold tolerance. Figure 2.2 shows a flow chart of Wexler’s algorithm.

Wexler (1988) claimed that since the method does not involve the explicit inversion of the Sensitivity Matrix the method is not subject to the problems of ill-conditioning. However, this method appears to need very many iterations to produce a reasonable image (Webster, 1990)

2.2.3 Newton-Raphson algorithm

In 1987, Yorkey et al (1987) proposed the Newton-Raphson iterative method to compute the conductivity distribution. The algorithm can be summarized as the following:

1) Inject a current into the domain of the imaged object $\Omega$ and record the potential $V_m$ on the boundary of the domain $\Gamma$.
2) Using FEM calculate the potential $V_0$ for the same injected current due to the initial conductivity $\sigma_0$.
3) Define the error functional $E(\sigma)$ such that:

$$E(\sigma) = \frac{1}{2}[V_m - V_0]^T [V_m - V_0]$$  \hspace{1cm} (2.28)

where $T$ is the matrix transpose.

The minimization of this error with respect to the conductivity gives:

$$E'(\sigma) = [V_m]^T [V_m - V_0] = 0$$  \hspace{1cm} (2.29)

where $[V_m]_{ij} = \frac{\partial F_i}{\partial \sigma_j}$ and is called the Jacobian Matrix.

Yorkey (Yorkey et al, 1987) calculated the above by applying Taylor’s expansion and ignoring the non-linear terms in order to obtain a correction factor for the conductivity at the $k^{th}$ iteration given by
\[ \Delta c^k = - \left[ \{V'(c^k)\}^T \{V'(c^k)\} \right]^{-1} \left[ \{V'(c^k)\}^T \{V(c^k) - V_0\} \right] \] (2.30)

The updated conductivity distributions becomes:

\[ c^{k+1} = c^k + \Delta c^k \] (2.31)

Figure 2.3 summarises the different stages in that algorithm.

In this algorithm, the Jacobian Matrix is recalculated at each iteration and thus an efficient FEM implementation is required to compute the Forward Problem as well as the Jacobian Matrix. Since the matrix \( \left[ \{V'(c^k)\}^T \{V'(c^k)\} \right] \) is ill-conditioned, regularization techniques are required if images are to be reconstructed from real data measurement which will obviously include measurement noise. For that reason regularization techniques have been used to obtain more stable solutions such as the Tikhonov-regularization method. Since then several groups have adopted a single strategy based on the iterative Newton-Raphson method (Yorkey et al., 1987).
2.3 Noniterative algorithms

Noniterative (or Single-Step) algorithms are based on the principle of linearization of the Inverse Problem. Furthermore, it will also be shown in this section that the Forward Problem needs to be solved once. The advantage of this approach is the algorithm simplicity and the speed of the reconstruction. But the disadvantage of this approach is less accuracy. Figure 2.4 summarizes the different steps involved in the noniterative algorithm.

![Diagram of noniterative algorithm](image)

There are different image reconstruction algorithms based on the noniterative method (Morucci and Marsili, 1996). They are all dependent on the principle of sensitivity relationship. In this section, the sensitivity relationship is firstly described, then the linearized sensitivity relationship will be described, followed by a discussion on few al-
algorithms developed during the 1980s which use the linearized sensitivity relationship. Finally, the well known Sheffield Filtered Back-Projection algorithm will be presented.

2.3.1 The Sensitivity Relationship

By using Gauss’s divergence theorem in an arbitrary boundary region \( \Omega \), whose boundary \( \Gamma \) is a piecewise smooth surface, then for a continuous vector function \( H \) in \( \Omega \), we have:

\[
\int_{\Gamma} H \cdot ds = \int_{\Omega} \nabla \cdot H \, dv
\]  
(2.32)

where \( ds \) is the outward facing normal unit vector to the boundary.

By taking any two scalar functions \( \Phi \) and \( \Psi \) and replacing \( H \) by \( \Phi \nabla \Psi \) in the previous equation, we have

\[
\int_{\Gamma} \Phi \nabla \Psi \cdot ds = \int_{\Omega} \Phi \nabla \cdot (\nabla \Psi) \, dv + \int_{\Omega} \nabla \Phi \cdot \nabla \Psi \, dv
\]  
(2.33)

Substituting \( \sigma \nabla \Psi \) for \( \nabla \Psi \), and defining \( \Psi \) to be a solution of equation (2.21) such that the first term on the right hand side of the above equation is zero, we have:

\[
\int_{\Gamma} \Phi \sigma \nabla \Psi \cdot ds = \int_{\Omega} \sigma \nabla \Phi \cdot \nabla \Psi \, dv
\]  
(2.34)

Suppose a pair of electrodes 1-2 are placed around the surface and conduct a current \( I_\psi \) into the region \( \Omega \), then a potential field \( \Psi \) is established within the region and on the surface \( \Gamma \). From Ohm’s law \( J = \sigma E \), \( \sigma \nabla \Psi \) is simply the negative of the current density. If \( \Omega \) is surrounded by a non-conducting region, the surface integral will be zero except at the electrode sites where current is passing into and out of \( \Omega \). The surface integral for this case is:
Similarly, if $\Phi$ is a solution to equation (2.21) when driving current $J_\Phi$ through electrodes 3-4 the surface integral is now equal to $I_\Phi \Phi_{34}$. If we assume unit current is applied in each case then:

$$g = \Phi_{12} = \Psi_{34} = \int_{\Omega} \sigma \nabla \Phi \cdot \nabla \Psi dv$$

(2.36)

If we drive unit current $I$ into a region $\Omega$ through electrodes 1-2, the voltage $g$ measured between electrodes 3-4 will be the same as the one measured between 1-2 if unit current $I$ were driven between 3-4. This is the statement given in 1971 by Geselowitz (Geselowitz, 1971), (Lehr, 1972).

Equation (2.36) is an important result in EIT as it describes the relationship between the measured boundary voltages and the conductivity distribution. However, this relationship is non-linear as $\nabla \Phi$ and $\nabla \Psi$ are also functions of the conductivity $\sigma$. From this theorem, we can approximate the sensitivity relationship by considering the general conductivity distribution and associated voltage measurements in terms of uniform and perturbed conductivity distributions:

$$\sigma = \sigma_u + \sigma_p$$

(2.37)

$$g = g_u + g_p$$

(2.38)

$$\nabla \Phi = \nabla \Phi_u + \nabla \Phi_p$$

(2.39)

$$\nabla \Psi = \nabla \Psi_u + \nabla \Psi_p$$

(2.40)

where the subscripts $u$ and $p$ are the uniform and perturbed distributions respectively.

Expanding equation (2.36) gives:
\[ g = \int_{\Omega} (\sigma_u + \sigma_p)(\nabla \Phi_u + \nabla \Phi_p) \cdot (\nabla \Psi_u + \nabla \Psi_p) dv \]  
\hspace{1cm} (2.41)

and using Barber’s derivation (1990) and assuming a constant current is applied to region \( \Omega \), this reduces to

\[ g = \int_{\Omega} \sigma_u \nabla \Phi_u \cdot \nabla \Psi_u dv - \int_{\Omega} \sigma_p \nabla \Phi_u \cdot \nabla \Psi_u dv + \int_{\Omega} \sigma_p \nabla \Phi_p \cdot \nabla \Psi_p dv \]  
\hspace{1cm} (2.42)

Since \( g_u = \int_{\Omega} \sigma_u \nabla \Phi_u \cdot \nabla \Psi_u dv \) and using equation (2.38), the perturbed voltage is given

\[ g_p = -\int_{\Omega} \sigma_p \nabla \Phi_u \cdot \nabla \Psi_u dv + \int_{\Omega} \sigma_p \nabla \Phi_p \cdot \nabla \Psi_p dv \]  
\hspace{1cm} (2.43)

By using the Barber’s identity (Barber, 1990):

\[ \int_{\Omega} \sigma \nabla \Phi_p \cdot \nabla \Psi_p dv = - \int_{\Omega} \sigma_p \nabla \Phi_u \cdot \nabla \Psi_p dv \]  
\hspace{1cm} (2.44)

Equation (2.43) now becomes:

\[ g_p = -\int_{\Omega} \sigma_p \nabla \Phi_u \cdot \nabla \Psi_u dv - \int_{\Omega} \sigma_p \nabla \Phi_u \cdot \nabla \Psi_p dv \]  
\hspace{1cm} (2.45)

But using the expansion of equation (2.40) for \( \nabla \Psi \), we have:

\[ g_p = -\int_{\Omega} \sigma_p \nabla \Phi_u \cdot \nabla \Psi dv \]  
\hspace{1cm} (2.46)

Equation (2.46) is the sensitivity relationship. If the conductivity perturbation \( \sigma_p \) is small, then \( \nabla \Psi_p \) is sufficiently smaller than \( \nabla \Psi_u \) to allow the second right term of equation (2.45) to be ignored. Our sensitivity relationship can then be approximated as:
\[ g_p = - \int_\Omega \sigma_p \nabla \Phi_u \cdot \nabla \Psi_u dv \]  
(2.47)

which is the linearized sensitivity relationship given by Barber (1990).

### 2.3.2 The Linearised Sensitivity Matrix

By splitting the region \( \Omega \) into a number of discrete elements of uniform conductivity, we can represent equation (2.47) as a set of linear equations. In matrix form this can be written as:

\[ g_p = Sc_p \]  
(2.48)

where

- \( g_p \) is a vector of the change in measurement data,
- \( c_p \) is the discretised conductivity vector,
- \( S \) is the sensitivity matrix whose coefficients are given by

\[ S_{ij} = - \int_{j^{th} \text{pixel}} \nabla \Phi_u \cdot \nabla \Psi_u dv \]  
(2.49)

where

- \( i \) corresponds to the \( i^{th} \) drive pair-receive pair electrode combination,
- \( j \) to the \( j^{th} \) element.

In the linear limit, we can recover the conductivity distribution \( c_p \) by inversion of the sensitivity matrix, thus:

\[ c_p = S^{-1} g_p \]  
(2.50)

By knowing the boundary measurement \( g_p \), we can therefore compute the perturbed conductivity \( c_p \). This result is valid for regions of any dimensionality and electrode arrangement, however the placement of electrodes and the configuration of the electrode pair will affect the stability of the matrix inversion (Avis, 1993) and hence image reconstruction performance.
In 1981, Yamashita and Takahashi (1981) used the finite element method to derive a linearized sensitivity matrix for a representation of the human thorax. They used regularized methods in order to form a stable inverse since the matrix was ill-conditioned. Still in 1981, Nakayama et al (1981) implemented a linearised iterative algorithm by assuming only small conductivity changes. They found regional variation in the accuracy algorithm using simulated reconstructions stopping at the first iteration.

In 1983, Sakamoto and Kanai (1983) computed the coefficients of the linearised sensitivity matrix by using an analytical method in order to reconstruct experimental data from a saline filled tank.

In 1985, Murai and Kagawa (1985) developed an iterative approach based on Geselowitz’s theorem where the change in transfer impedance $\Delta Z$ measured at the surface of a body is related to the conductivity change $\Delta c$ as:

$$\Delta Z = S\Delta c$$  (2.51)

where the sensitivity matrix $S$ is computed from the lead fields ($\nabla \Phi/I_e$) for a given conductivity distribution.

Comparing the values of the computed transfer impedance $Z$ to those which are measured $Z_m$ the change in transfer impedance becomes:

$$\Delta Z = Z_m - Z = S\Delta c$$  (2.52)

By inverting matrix $S$, $\Delta c$ is computed and used to update the conductivities values. The iteration is repeated until the conductivity is found. This method is similar to that implemented by Yorkey except that the conductivity change is measured.

In 1994, Morucci et al (1994) (1995) proposed a method called “Direct Sensitivity Matrix (DSM)”. In that method, a finite element method is used to compute the change in the boundary voltage for a small conductivity change in a single element. By applying
each drive/receive for each element, the full matrix is constructed. The coefficients of the DSM are found as

\[ S_{ij} = \frac{\partial C_i}{\partial V_j} \approx \frac{\delta C_i}{\delta V_j} \]  

(2.53)

where \( S_{ij} \) is a coefficient of the DSM for the \( i^{th} \) element and \( j^{th} \) drive-receive combination.

It must be pointed out that Morucci uses the reciprocal of each sensitivity coefficients. Furthermore, Morucci also only used the small coefficients of \( S \) and replaced the large coefficients with zeros. These rejected values correspond to the small voltage variations and are therefore the least reliable in terms of noise.

**The Sheffield Filtered Back-Projection algorithm**

Since the work presented in this thesis uses some concepts developed initially from this type of algorithm, this subsection will describe this algorithm in more detail than those presented above. This algorithm is probably the most used algorithm for medical applications since it is a fast and robust algorithm.

In 1983 Barber *et al* (1983) developed the initial algorithm and this was re-formulated in 1990 (Barber, 1990). The principle of this algorithm is based on the use of a sensitivity matrix derived from equation (2.50). The idea is that an image of perturbed conductivity distribution can be found if an accurate sensitivity matrix \( S \) can be determined and a stable inversion of \( S \) can be evaluated using regularization techniques. However, Barber and Brown (1988) showed that the reconstruction of the sensitivity matrix must be calculated with an identical geometry to those found during data collection since even small electrode misplacement can result in quite large errors in reconstructed image\(^3\). Based on that observation, they modified the reconstruction by normalising both sides of equation (2.48) by a reference dataset. This resulted in producing differential images (or dynamic

\(^3\) For industrial process tomography, this condition could be met, but not for medical applications.
images) rather than static images. This also implied that only the normalised change in conductivity is determined.

Barber chose to reconstruct normalised perturbed data such that each individual boundary voltage measurement $g_n$ is given by:

$$g_n = \frac{g_p}{g_u}$$  \hspace{1cm} (2.54)

where  

$g_u = c_u \int_{\Omega} \nabla \Phi \cdot \nabla \Psi dv$ is the uniform boundary voltage, 

$c_u$ is the uniform conductivity distribution.

The errors introduced in $g_p$ and $g_u$ as a result of deviations from the expected geometry are likely to be similar and will tend to cancel out or at least will be substantially reduced.

The above equation (2.54) can be re-written as

$$g_n = \frac{\mathbf{G}^{-1} g_p}{c_u}$$  \hspace{1cm} (2.55)

where $\mathbf{G}$ is a diagonal matrix whose non-zero elements are:

$$G_{ii} = \int_{\Omega} \nabla \Phi_{ii} \cdot \nabla \Psi dv$$  \hspace{1cm} (2.56)

where $i$ refers to the particular drive-receive pair combination.

Using $g_p = \mathbf{S} c_p$, equation (2.55) can be written as:

$$g_n = \frac{\mathbf{G}^{-1} \mathbf{S} c_p}{c_u}$$  \hspace{1cm} (2.57)

writing $\mathbf{G}^{-1} \mathbf{S} = \mathbf{F}$ and $\frac{c_u}{c_n} = c_n$, equation (2.57) becomes

\footnote{It must be pointed out that the $g_u$ is usually measured. However, it could be calculated analytically or numerically. This will be explained later.}
The matrix $F$ now describes the normalised linear sensitivity relationship between normalised changes in the boundary voltages and the normalised changes in conductivity. If a stable inversion of the normalised sensitivity matrix can be found, then images of normalised change in conductivity can be obtained from the normalised boundary data. However, it is unlikely that $F$ will be any better conditioned than $S$ and inversion will still be difficult. It is for this reason that Barber et al decided to approximate the inversion of $F$ by implementing the back-projection scheme.

As mentioned in chapter 1, this arose from the idea of modeling the filtered back-projection image reconstruction process associated with X-ray where electric beams are used rather than beams of X-ray. Indeed, in CT or other related imaging modalities, the measured values (e.g. the logarithm of the incident and received beams in CT) are back-projected via straight lines since only the elements on that line cause the changes in the measured beam. The term back-projection is used since the measured data are added to all the elements that are on the straight line that connects the X-ray source and the receiver. In addition, the pixel values are appropriately weighted depending on the position of the element in the image area. However, in EIT the straight line method cannot be used since the changes in all the elements in the object affect all the measurements. Therefore, the idea is to back-project the normalised boundary measurements along the equi-potential loci (formed from injected current through a dipole on the surface) rather than back-projecting along straight lines.

For each reconstructed element and for each current pattern, the measurements made on the boundary are back-projected in this element. Therefore, in matrix notation, the back-projection operator can be written as

$$g_n = F c_n \quad (2.58)$$
\[ c_m = Bg_n \] 

(2.59)

where

- \( B \) is the back-projection operator \((m \times n)\) matrix of back-projection weight,
- \( g_n \) is the normalised change in \( n \) boundary voltage measurements,
- \( c_m \) is a conductivity vector of \( m \) pixels.

The lines that are used for back-projection are the equipotential lines resulting from each current injection pair since for a given element the maximum measurement sensitivity can be found by the intersection of the equipotential line going through that element and the boundary (Breckon and Pidcock, 1987) (Santosa and Vogelius, 1990). Adjacent drives are approximated by placing dipoles on the surface. In a two dimensional plane these form curved lines which extend from the dipole to the boundary and are orthogonal to the current path. The equipotential lines are not known due to the unknown conductivity. Therefore, they are approximated based on the assumption that the object is circular having uniform conductivity.

Barber and Brown (Barber and Brown, 1986) use Conformal Transformation as an efficient method of determining the lines between the dipole and the voltage measurement electrodes rather than calculating the equipotential loci analytically. Since non-central objects are probed by a non-uniform angular distribution of currents, the same technique can be used to weight the back-projection in order to avoid assigning each pixel lying along the equipotential the same value. By weighting the back-projection coefficients according to the angular density of equipotentials, an isotropic reconstruction can be obtained (Barber and Seagar, 1987). Furthermore, other assumptions are made:
1) The object being imaged is two dimensional.

2) There are no internal current sources.

3) The medium is isotropic.

4) The boundary of the region is circular.

5) The electrodes are arranged with equidistant spacing around the boundary.

6) The initial conductivity distribution is uniform.

7) The change in conductivity is small.

The effects of violating these conditions have been studied in detail by (Avis, 1993), (Barber and Brown, 1990), (Kiber, 1991). Barber and Brown (1990) and Kiber (1991) showed that assumptions three and four could be met with the use of conformal transformation since it transforms non-circular 2D region to a circular 2D region.

The back-projection algorithm typically uses 16 dipole positions per adjacent drive pair (Avis, 1993). Good examples of how the back-projection works can be found in the following literature (Barber, 1992), (Avis, 1993), (Metherall, 1998), (Rigaud et al, 1996).

All images formed by the back-projection technique are blurred and contain star or spoke like artifacts (Ott et al, 1992). These can be removed by filtering the image (or more innovatively filtering the data prior to back-projection). Furthermore, image reconstruction in EIT is point spread variant which means that the resolution of the reconstructed image varies with position in the image. Therefore, the restoration filter must take that into account.

The filtered back-projection image reconstruction algorithm is:

---

A conformal transformation between two spaces is one in which the angles between two lines are preserved. It transforms a complex problem into a space in which the resulting geometry permits an easier solution.
\[ c_n = EBg_n \]  \hspace{2cm} (2.60)

where

\( E \): the matrix representation of the empirical restoration filter,
\( c_n \): approximation of the normalised changes in conductivity distribution.

Santosa and Vogelius (1990) demonstrated that the back-projection operation is an approximation to the generalised inverse Radon Transform. Based on that, Barber (1990) derived a new formulation for the back-projection reconstruction algorithm where the boundary data are transformed such that when they are back-projected in the usual manner, it leads to the correct image rather than filtering the image after back-projection.

Suppose that the transformed boundary data are described by the vector \( g'_n \), then the back-projection results in the correct image as :

\[ c_n = Bg'_n \]  \hspace{2cm} (2.61)

Using equation (2.58) the above equation becomes :

\[ g_n = FBg'_n \]  \hspace{2cm} (2.62)

Now \( g'_n \) is

\[ g'_n = (FB)^{-1}g_n \]  \hspace{2cm} (2.63)

Using equation (2.61) and equation (2.63), the normalised conductivity becomes :

\[ c_n = B(FB)^{-1}g_n \]  \hspace{2cm} (2.64)
The filtration stage of the “filtered back-projection” method is achieved by pre-multiplication of the measured boundary data and is followed by the back-projection. Furthermore, the need for an empirically derived filter has therefore been eliminated. Barber and Brown (Barber and Brown, 1990) observed a significant improvement in the image resolution compared to existing image filtration but noted “ringing” artefacts were also introduced.

Furthermore, the matrix $B$ in the product $FB$ needs not to have the same structure as the leading matrix $B$ (Avis, 1993). Therefore, the leading matrix $B$ can be tailored to the desired pixellation scheme which is appropriate for the display resolution. This method has the advantage of also incorporating *a-priori* information into the reconstruction algorithm and thus addresses the assumption of a non-uniform reference conductivity distribution. Nevertheless, incorporating that information directly into the $B$ matrix by back-projecting along the true equipotential loci was unsuccessful (Avis, 1993). Avis and Barber (1995) modified the forward projection matrix $F$ in order to incorporate the *a-priori* information using a FEM approach. They have applied it to the head where *a-priori* information consisted of regions of varying conductivity used to represent the scalp, skull and brain tissues. They concluded that they could successfully reconstruct images.

In 1994, Avis and Barber (1994) developed a generalised image reconstruction algorithm based on the filtered back-projection method where the weighting factors required for constructing the back-projection matrix could be found for any bipolar electrode configuration using bipolar transformation. This allows the possibility of using non-adjacent data collection strategies which may offer improved signal-to-noise ratios (Smith, 1990).

---

6 Bipolar term was introduced by Avis (1993) to distinguish investigations using non-adjacent drive configurations, from the dipole approximation used for the adjacent pair format.
2.4 Other Image Reconstruction Algorithms

There are a few algorithms which cannot easily be classified into the two categories presented above, namely iterative and noniterative categories. This is the reason why they are presented in this section.

2.4.1 NOSER algorithm

This algorithm was proposed by the Rensselaer Polytechnic Institute research team (Cheney et al., 1990), (Isaacson et al., 1992), (Isaacson et al., 1992), (Isaacson and Epic, 1992), (Cheney et al., 1999). This algorithm uses a single-step of the Newton-Raphson technique which was described in the iterative section. Despite being based on the Newton-Raphson approach, this is a linear technique since it uses only one iteration of the Newton-Raphson algorithm to reconstruct images. Furthermore, the “optimal current” configuration is used in this algorithm to obtain the best distinguishability (see chapter 1).

This algorithm is fast at reconstructing images. However, it is not very accurate. Cheney and Isaacson (1995) have presented encouraging results of synthetic lungs manufactured from agar and suspended in a saline filled tank. However, in-vivo images from a human thorax appear rather distorted, as the lung regions can only be identified close to the boundary (Edic et al., 1995), (McLeod et al., 1996). One of the reasons could be the artefacts associated with the electrodes.

2.4.2 Layer Stripping algorithm

This algorithm solves directly the full non-linear problem without any iterations. The idea is based on finding the conductivity for different concentric layers (Cheney et al., 1991), (Cheney and Isaacson, 1992), (Cheney and Isaacson, 1995), (Somersalo et al., 1992). This technique involves finding the conductivity at the boundary of a disc or a sphere (of radius $r_0$) and using this information to estimate the internal voltage at a radius ($r_0 - \Delta r$).
Using this data, the process is repeated and thus the conductivity distribution is obtained layer by layer.

In other words, assume that a very high spatial frequency current pattern is applied into a body. This means that the current will not penetrate very deeply into the body and is only influenced by the conductivity distribution near the boundary. Therefore, the conductivity is estimated by the boundary voltages for a thin layer near the boundary. Once the conductivity in the outermost layer is approximately found, the outcome of the similar experiment, if the known layer was stripped away, can be computed. The conductivity of this second layer is approximated and stripped away, and this is repeated through the whole body. Thus, the conductivity of the whole body can be estimated. Unfortunately, there are no results with experimental data due to the fact that many simplifications in the modelling of the electrodes and errors in the reconstruction on the outermost layers propagates and pollutes the reconstruction in the interior regions (Isaacson et al, 1992).

### 2.4.3 Other approaches

**Probabilistic approach**

Only a few probabilistic approaches have been used in EIT image reconstruction due to the fact that the conductivity distribution in EIT has always been seen as deterministic. The probabilistic approach will not be reviewed in this thesis since it is beyond the scope of this thesis. However, the reader is referred to the Vaukhonen’s PhD thesis (1997) in which he presents the probabilistic approaches and demonstrates the connection with the deterministic approaches.

**Neural Network approach**

In 1994, a neural network algorithm was proposed by Adler and Guardo (1994). The neural network was trained by giving a set of inputs, i.e. the normalised change in the
voltage measurements including certain amount of noise and the corresponding outputs, i.e. the changes in the conductivities to the network. After the learning process the image was reconstructed.

2.5 Multi-Frequency EIT

During the mid 80s, the main characteristics of EIT were:

1) Only the conductivity distribution was imaged since quasi-static conditions are applied (see chapter 2).

2) As iterative algorithms are difficult for performing in-vivo measurements, noniterative algorithms producing dynamic images (difference between two datasets recorded at two different times) were used.

3) An alternating current (AC current) operating at a single frequency (low frequency) was used for injecting the current into the system.

Furthermore, the electrical spectroscopy community showed that significant differences in the electrical conductivity of various tissues as a function of the frequency existed (Jossinet et al, 1985).

Based on these observations, several research groups started to think that it could be possible to generate dynamic images not longer in the time domain but rather in the frequency domain. Indeed, dynamic images are generated by two dataset recording at two different times where the first dataset is used for normalising the second dataset. This process allows the reconstruction algorithm to be less sensitive to noise and to electrode placements. Therefore, it was thought that a similar process could be done in the frequency domain where the two datasets will be recorded at two different frequencies. The dynamic images created in the frequency domain would be called dual-frequency images since they are generated at a single time. One of the first research groups to propose to use the frequency domain rather than the time domain for generating images was Griffiths and Ahmed (1987). Indeed, they proposed to use the variation differences in the
electrical conductivity of various tissues as a function of the frequency in order to form images. Their method (described in the next section) consisted of taking as reference the data (real part of the impedance) measured at the low frequency with the assumption that the imaginary part could be ignored at that frequency and by back-projecting the ratio, either of the real part measured at the higher frequency to the real part measured at the low frequency or of the imaginary part measured at high frequency to the real part measured at low frequency.

The electrical spectrum community demonstrated also that it was possible to determine the state of tissue (healthy or diseased) and to be able to differentiate them according to the frequency (Jossinet, 1998). The process of doing that was the use of the Cole-Cole plot at different frequencies (see chapter 1). Based on that and the fact that images could be now generated in the frequency domain, several research groups in EIT started to think that it could be possible to generate images at different frequencies and from those images with the use of the Cole-Cole plot to generate an image (named parametric image) which will represent the state of different tissues being imaged. In other words, it was not only possible to relate the different tissues appearing in the image with their state (healthy or diseased), but also to determine which pixels belong to which tissue. The process of generating parametric images will be described later in this section.

2.5.1 Dual-frequency images

From Griffiths (1987), Jossinet et al (1985) and Riu et al (1993), dual-frequency images are formed (Rigaud et al, 1996) as follows:

1) A set of measurements is made at low frequency \(f_1\) as a reference. The use of a low frequency is to ensure that only the real part of the impedance is measured since the imaginary part of the impedance is assumed to be small at low frequency.

2) Another set of measurements is taken at a higher frequency \(f_2\)
3) The ratio, which is either the real part of the impedance at $f_2$
over the real part of the impedance at $f_1$ or the imaginary part of the
impedance at $f_2$ over the real part of the impedance at $f_1$, is back-projected.

4) The image reconstruction process results in an amplitude image in the first case and
a phase image in the second case.

In this method, the images reconstruct impedance variation over the frequency rather
than over the time. The frequency range was defined between 10 kHz-1MHz in order
to be able to identify dielectric properties of biological tissues and to account for the
limitations in the hardware. Figure 2.5 shows an example of dual-frequency images
where a set of differential thoracic images are obtained at eight different frequencies
from 9.6 kHz to 1.2 MHz (Brown et al, 1994)

Figure 2.5. Example of quasi-static images from (Brown et al, 1994)
2.5.2 Parametric images

Jossinet and Trillaud (1992) defined a relationship between the impedance tomography and impedance spectrometry. Their idea was to perform complex tomography measurements to calculate the parameters which are used in the Cole-Cole plot (McAdams and Jossinet, 1995) where the Cole-Cole equation (Cole and Cole, 1941) is

\[ Z = R_\infty + \frac{R_0 - R_\infty}{1 + (j2\pi f/f_c)^\alpha} \]

where

- \( R_0 \): the resistance at low frequency,
- \( R_\infty \): the resistance at high frequency,
- \( f \): the frequency,
- \( f_c \): the characteristic frequency,
- \( \alpha \): the dispersion parameter.

Later, Griffiths and Jossinet (1994) proposed a method in which the Cole plot (see figure 2.6) is obtained from multi-frequency impedance images, provided that reference data have a negligible imaginary part.

Figure 2.6. Reconstruction of parametric images using Cole plot.
Adapted from Casas et al (2000).
An example of parametric images is also given by Brown (1995) where only the real component of the impeditivity is used (see figure 2.7). In this example, the average parametric images for 12 normal subjects obtained at a maximum expiration and maximum inspiration is obtained. In this method, the image is reconstructed in a similar fashion to dynamic images in EIT. But this time different boundary voltages are recorded at different frequencies. At the end, each pixel has a number of conductivity change value for each frequency. For each pixel, the different values along the frequencies range are plotted on a Cole-Cole plot. The parameters R, C, FR and S are determined by the Cole’s electrical circuit model.

2.6 Discussion

This chapter has described how EIT images are generated. The first step in the image reconstruction algorithm is the Forward Problem which is solved either mathematically
by the use of analytical (or numerical) tools or experimentally by the use of a tank. The Forward Problem is a crucial step in the image reconstruction algorithm since it models the behaviour of the reality. Although it can be solved experimentally by the use of a tank, it is usually solved mathematically since it allows the different parameters (topology and electrodes) to be changed easily.

Solving the Forward Problem mathematically means solving Poisson’s (or Laplace’s) equation with correct boundary conditions and for a certain topology (see section 2.1). Poisson’s (or Laplace’s) equation can be used since Maxwell’s equations reduce to them on account of application of the quasi-static conditions (see section 2.1). However, the quasi-static conditions imply the use of low frequencies.

Parametric images are generated from images recorded at different frequencies and dual-frequency images are generated from two datasets recorded at two different frequencies. To date, instruments can use frequencies above the megahertz and, therefore, it is important to know at which frequency (according to the tissue being analysed) these quasi-static conditions will fail. Otherwise, there could be an anomaly between the image reconstruction algorithm solving the Forward Problem using the quasi-static conditions and the experimental measurements. Therefore, chapter 7 will investigate the behaviour of the quasi-static conditions for certain tissues at different frequencies and for a certain topology.
Chapter 3

Methods for Modelling the Forward Problem

As described in chapters 1 and 2, the solution to the Forward Problem for both iterative and noniterative algorithms is very important. Furthermore, the accuracy of the Forward Problem is essential. A solution to the Forward Problem can be obtained either analytically or numerically. Numerical solutions are often chosen because the object to be imaged in EIT (such as the human body) cannot be approximated by a simple geometrical shape. As a consequence, numerical techniques (such as Finite Element method (FEM) or Boundary Element Method (BEM)) are often chosen for solving the Forward Problem instead of the analytical solution. This is especially true for the iterative algorithms since they are very sensitive to the shape of the object being imaged as they attempt to reconstruct the absolute value of the conductivity. However, analytical solutions can still be used with good effect when noniterative algorithms are applied since they are less sensitive to the shape of the object since they reconstruct only differential images. Furthermore for some applications of EIT which are concerned with imaging regular shaped objects, such as in Industrial Process Tomography (IPT), analytical solutions can provide the basis for both noniterative and iterative algorithms. They also have some significant advantages compared to a numerical method.

This chapter starts by discussing why analytical solutions are useful in EIT. Then, it will be shown how noniterative algorithms can use an analytical solution to the Forward Problem for generating images. Finally, the analytical tools which will be used for deriving an analytical solution to the Forward Problem in this thesis, will be presented.
3.1 Analytical solutions in EIT

It is well known that analytical solutions have some limitations since

1. They model only simple geometrical shapes such as a finite circular cylinder, a sphere or an elliptical cylinder.
2. Only simple electrode models can be applied.
3. There are often problems associated with the convergence to a solution in the computational implementation.

Nevertheless, analytical solutions have some significant advantages compared to a numerical solution like the FEM (or BEM) since:

1. They do not depend on the type of elements used for meshing the object.
2. They do not depend on the degree of the interpolation function used in the shape functions.
3. They do not depend on the type of solver employed unlike the FEM (or BEM) methods do.
4. When the third dimension is addressed, the FEM (or BEM) can become more computationally expensive than an analytical solution.

Furthermore, an analytical solution in EIT can also

5. Provide a basis for noniterative EIT image reconstruction algorithms.
6. Be used for validating a FEM (or BEM) solution.

It can be seen from the advantages (1-6) associated with an analytical solution that analytical solution can still provide useful results for EIT as well as for IPT. This is the reason why the first aim of this thesis will extend analytical solutions used in the past for two dimensional cases (Avis, 1993) to the third dimension.
3.2 Reconstruction of EIT images using analytical solutions

As mentioned at the beginning of this chapter, noniterative algorithms allow an analytical solution to the Forward Problem to be employed in the image reconstruction process. This is mainly due to the normalisation of the data which appear in noniterative algorithms and therefore make the algorithm less sensitive to the position of the electrodes and the shape of the object being imaged allowing the Forward Problem to be approximated by non numerical methods. Since the second aim of this thesis is to be able to reconstruct three dimensional images from an analytical solution to the Forward Problem, this section describes how images can be reconstructed using analytical methods.

In the linearised sensitivity relationship, the perturbed conductivity was given by equation (2.48) repeated here for convenience.

\[
G_{125}G_{c82}G_{G39}G_{G55}G_{G83}G_{G39}
\]

\[
(3.1)
\]

where

\[
G_{G125}G_{c82}G_{G83}G_{c82}G_{G55}G_{c51}G_{c129}G_{G125}G_{c82}
\]

\[
(3.2)
\]

By knowing the boundary measurement \(G_{G125}G_{c82}G_{G99}\) the perturbed conductivity \(G_{G83}G_{c82}G_{G39}G_{G55}G_{c51}G_{c129}G_{G125}G_{c82}\) can be computed. Furthermore, not only the sensitivity matrix can be computed by a numerical
method, but it can also be computed by an analytical method since the sensitivity matrix is calculated for a uniform distribution.

Although it is possible to reconstruct an image using equation (3.3) with an analytical solution to the Forward Problem, the reconstructed image will not be very accurate for a medical application since the placement of electrodes and the configuration of the electrode pair will affect the rank of the sensitivity matrix and the stability of the matrix inversion (Avis, 1993) and hence image reconstruction performance\(^7\).

Nevertheless, the problem concerned with the placement of the electrodes and the shape of the object being imaged can be significantly reduced by normalising the data using the normalisation equation introduced in chapter 2 and shown here for convenience.

\[
g_n = F c_n \tag{3.4}
\]

where

\[
F = G^{-1} S \quad \text{(forward matrix)},
\]

\[
G_{ii} = \int_{\Omega} \nabla \Phi_u \cdot \nabla \Psi_u dv \quad \text{(diagonal matrix)},
\]

\[
S_{ij} = - \int_{\Omega} \nabla \Phi_{uj} \cdot \nabla \Psi_u dv \quad \text{(sensitivity matrix)},
\]

\[
c_n = \frac{c_p}{c_u} \quad \text{(normalised conductivity distribution)},
\]

\[
c_p : \text{perturbed conductivity distribution},
\]

\[
c_u : \text{uniform conductivity distribution},
\]

\[
g_n = \frac{g_p}{g_u} \quad \text{(normalised boundary voltage)},
\]

\[
g_p : \text{perturbed boundary voltage},
\]

\[
g_u : \text{uniform boundary voltage}.
\]

\(^7\) For an IPT application, the reconstructed image would be better since the placement of the electrodes is known and the shape of the object being imaged is simple.
Since $G$ and $S$ are computed for the uniform case, an analytical solution can be used. By using the above equation with an analytical solution to the Forward Problem, dynamic images for medical application can be reconstructed with the constraint that the parameters of the simple geometrical shape used for modelling the shape of the object being imaged have not changed. As the data are normalised, the effects due to a simple geometrical shape and due to the movement of the electrodes are reduced.

The above equation combined with noniterative algorithms provides a significant advantage compared to iterative algorithms as the computation of $S$ and $G$ must only be performed once as long as the geometrical shape used as a model does not change.

A good example of the use of noniterative algorithms used for reconstructing images involving non numerical methods is the work done by Metherall (Metherall, 1996). He produced full three dimensional images of the human thorax using a $1/r$ model (Witsoe, 1967) as a solution to the Forward Problem. This expression computes the potential as:

$$\Phi = \frac{1}{r_1} - \frac{1}{r_2}$$

(3.5)

where $r_1$ is the distance between the point on which the potential is computed and the first electrode of the electrode pair driving the current. $r_2$ is the distance between the point on which the potential is computed and the second electrode of the electrode pair driving the current.

It is based on the electrostatic potential due to a point charge $Q$. By using that expression and the Sensitivity method, three dimensional images were produced. They used that expression as a pseudo analytical solution to the Forward Problem for three reasons:

1. Finding a full analytical solution to the three dimensional Forward Problem is difficult (Barber and Brown, 1984),

2. The $1/r$ expression allows the computation of the Sensitivity Matrix to be fast,
3. The $1/r$ expression allows the computation of the Sensitivity Matrix in three dimensions on modest computational platforms.

However, this expression models the potential distribution for an infinite half space. Nevertheless, they showed very clearly that they could reconstruct full three dimensional images using the sensitivity method with only a pseudo-analytical solution to the Forward Problem since *in-vivo* images of the human thorax were presented. This illustrates the robustness of dynamic images as that expression is a gross approximation.

Based on this, it will be shown later in the thesis that a full three dimensional analytical solution to the Forward Problem for finite right circular cylinder can be derived and images can be reconstructed. Furthermore, some comparisons between that full analytical solution and the $1/r$ model will be presented in chapter 4.

### 3.3 Green’s Theorem, Delta Function and Green’s Functions

To understand how the boundary value problems have been solved analytically in this thesis, the Green’s theorem will be introduced (Smythe, 1989).

#### 3.3.1 Green’s Theorem

Let $V$ be a three-dimensional volume bounded by a surface $S$. A point inside this volume is located by the radius vector $r$. Inside the volume $V$ we have two unknown functions $\Psi(r)$ and $\Phi(r)$ which are finite and continuous along with their first and second partial derivatives. Given these conditions the Green’s theorem applies:

$$
\iint_V (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) dV = \iint_S (\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n}) dS \quad (3.6)
$$
where \( \frac{\partial f}{\partial n} \) is the derivative with respect to the outward normal (the rate of change of the function in the direction perpendicular to the surface).

If we assume that these two functions also satisfy the homogeneous Poisson’s equation on the surface and in the volume:

\[
\nabla^2 \Phi = 0 \quad (3.7)
\]
\[
\nabla^2 \Psi = 0 \quad (3.8)
\]

The integrand on the left hand side of equation (3.6) vanishes and equation (3.6) becomes:

\[
\int \int_S (\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n}) dS = 0 \quad (3.9)
\]

Thus if the functions \( \Psi \) and \( \Phi \) have no singularities within or on the surface \( S \), then equation (3.9) must be satisfied. This equation forms the basis for the derivation of the Poisson integral equation.

### 3.3.2 Delta Function

Before introducing the concept of Green’s functions, another function known as the delta function (Roach, 1982) must also be presented since it is often used in conjunction with the Green’s function for solving boundary value problems.

Let’s define \( L \) to be a linear, ordinary differential operator acting on the space of function \( u(x) \). By defining \( L^{-1} \) to be the inverse operator of \( L \) such that \( LL^{-1} = L^{-1}L = I \) and by assuming that \( L^{-1} \) is an integral operator with kernel \( K(x,t) \) such that:

\[
L^{-1}u(x) = \int K(x,t)u(t)dt \quad (3.10)
\]
Then, formally at least, \( u(x) \) can be re-written as

\[
    u(x) = I u(x) = LL^{-1} u(x) = L \int K(x, t) u(t) dt
\]

(3.11)

Since \( L \) is a differential operator with respect to the variable \( x \),

\[
    u(x) = \int LK(x, t) u(t) dt = \int g(x, t) u(t) dt
\]

(3.12)

where \( g(x, t) = LK(x, t) \).

Now, if this result is to be true for all continuous functions \( u \), then

\[
    \begin{cases}
        g(x, t) = 0 & x \neq t \\
        u(x) & x = t
    \end{cases}
\]

To ensure that, Dirac replaced \( g(x, t) \) by a new function called the delta function (\( \delta \)) (Dirac, 1947). Equation (3.12) must be re-written as

\[
    u(x) = \int \delta(t - x) u(t) dt
\]

(3.13)

where \( \delta(x) = 0 \) if \( x \neq 0 \).

The delta function has the property of being equal to 0 except at the origin where it becomes infinite in such a way that ensures \( \int_{-\infty}^{+\infty} \delta(x) dx = 1 \). It must be stressed that this function does not behave like a normal function. By extending the above relationship to three dimensions, it can be demonstrated that the delta function has the following property:

\[
    \iiint_V u(r) \delta(r - r_0) dV = \begin{cases}
        u(r_0) & r_0 \text{ in } V \\
        0 & r_0 \text{ not in } V
    \end{cases}
\]

(3.14)
The delta function with the above properties is often employed in the solution of boundary value problems.

### 3.3.3 Green’s Functions

A systematic way of obtaining a solution to linear boundary-value problems is the use of Green’s Functions which are also known as source functions or influence functions. They form the link between the differential and integral formulations. Furthermore, they offer a solution for dealing with source terms in a partial differential equation. The principle of Green’s function technique, is detailed below.

The principle of the Green’s function technique (Roach, 1982) can be explained by considering the following boundary value problem:

\[
\nabla^2 \Phi = g \quad \text{in} \quad R \\
\Phi = f \quad \text{on} \quad B
\]

where

- \( g \) is a source term,
- equation (3.15) is the Dirichlet’s boundary condition with \( f \) being a source on the boundary,
- \( R : \text{Region, } B : \text{Boundary} \).

Furthermore, suppose that there is a solution to the following boundary value problem:

\[
\nabla^2 G = \delta(r - r_0) \quad \text{in} \quad R \\
\text{and choosing } G = 0 \quad \text{on} \quad B
\]

where

- \( G \) is the Green’s function,
- \( \delta(r - r_0) \) is an impulse function,
- the Dirichlet’s boundary condition is homogeneous,
- \( R : \text{Region, } B : \text{Boundary} \).
By letting \( \Phi \) be the required solution to our partial differential equation (Poisson’s equation (Laplace’s equation)) and \( \Psi \) being the Green’s function, then replacing them into the equation (3.6) (Green’s theorem):

\[
\iiint_V (\Phi \nabla^2 G - G \nabla^2 \Phi)\,dV = \iint_S (\Phi \frac{\partial G}{\partial n} - G \frac{\partial \Phi}{\partial n})\,dS
\]

(3.19)

where
\( \Phi \) being a function,
\( G \) being the Green’s function,
\( \frac{\partial}{\partial n} \) is the derivative with respect to the outward normal,

Replacing the source term, the impulse function and the boundary conditions at the right positions in the space, equation (3.19) becomes

\[
\iiint_V (\Phi(r_0)\delta(r - r_0) - G(r, r_0)g(r_0))\,dV = \iint_S f(r, r_0)\frac{\partial G(r, r_0)}{\partial n}\,dS
\]

(3.20)

where
\( \Phi(r_0) \) being the solution at \( r_0 \),
\( g(r_0) \) being the source term at \( r_0 \),
\( G(r, r_0) \) being the Green’s function and representing the field at the observation point \( r \) caused by a unit point source point \( r_0 \),
\( f \) being the Dirchlet’s boundary conditions
\( \frac{\partial}{\partial n} \) is the derivative with respect to the outward normal.

By using equation(3.14), the above equation becomes

\[
\Phi(r) = \iiint_V G(r, r_0)g(r_0)\,dV - \iint_S f(r, r_0)\frac{\partial G(r, r_0)}{\partial n}\,dS
\]

(3.21)

Thus, the field at \( r \) by a source distribution \( g(r_0) \) is the integral of \( g(r_0)G(r, r_0) \) over the range of \( r_0 \) occupied by the source.
By knowing the Green’s function and by knowing its Neumann’s boundary conditions, it is possible to find the solution $\Phi$. Therefore, the solution centres around the construction of the approximate Green’s function.

### 3.3.4 Eigenfunction Expansion

There are three main methods for finding the Green’s function namely the method of Images, the method of Eigenfunction Expansion and the Operator method (Roach, 1982). The method of Images is mostly concerned with two dimensional problems and is based essentially on the construction of the Green’s functions for a finite domain from that of an infinite domain. The disadvantage of this method is that it can be applied only to problems with simple boundary geometries. Although, the Operator method can be used for deriving an analytical solution, it has the disadvantage to be longer than the Eigenfunction Expansion method in terms of derivation for deriving the potential different forms of a solution.

In this thesis, the method of Eigenfunction Expansion will be used for deriving the Green’s function for our boundary value problem. The main concept behind this method is to represent the Green’s function by (a set of) a series of orthonormal functions that satisfy the boundary conditions associated with the differential equation. To illustrate the method, the following example is presented in this section. Suppose that we want to find the Green’s function for the following boundary value problem:

\[
\begin{align*}
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + k^2 \Phi &= 0 \\
\frac{\partial \Phi}{\partial n} &= 0
\end{align*}
\]  

And let’s represent the Eigenfunctions and Eigenvalues of equation (3.22) that satisfy equation (3.23) by $\phi_j$ and $k_j$ such that:
Methods for Modelling the Forward Problem

\[ \nabla^2 \phi_j + k^2 \phi_j = 0 \]  \hspace{1cm} (3.24)

with \( \phi_j \) forming an orthonormal set:

\[ \int_S \phi_j^* \phi_i \, dx \, dy = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \]  \hspace{1cm} (3.25)

where the asterix (*) denotes complex conjugation.

Assuming that \( \phi_j \) forms a complete set of orthonormal functions, \( G(x, y; x_0, y_0) \) can be expanded in terms of \( \phi_j \) i.e,

\[ G(x, y; x_0, y_0) = \sum_{r=1}^{\infty} a_j \phi_j(x, y) \]  \hspace{1cm} (3.26)

Since the Green’s function must satisfy

\[ (\nabla^2 + k^2)G(x, y; x_0, y_0) = \partial(x - x_0)\partial(y - y_0) \]  \hspace{1cm} (3.27)

Substituting equations (3.26) and (3.24) into equation (3.27),

\[ \sum_{j=1}^{\infty} a_j (k^2 - k_j^2) \phi_j = \partial(x - x_0)\partial(y - y_0) \]  \hspace{1cm} (3.28)

Multiplying both sides by \( \phi_j^* \) and integrating over the region \( S \) gives

\[ \sum_{j=1}^{\infty} a_j (k^2 - k_j^2) \int_S \phi_j \phi_j^* \, dx \, dy = \phi_j^*(x_0, y_0) \]  \hspace{1cm} (3.29)

By imposing the orthonormal property in equation (3.29) leads to
Thus the Green’s function of our problem is

\[ a_i(k^2 - k_j^2) = \phi_j^*(x_0, y_0) \implies a_i = \frac{\phi_j^*(x_0, y_0)}{(k^2 - k_j^2)} \quad (3.30) \]

Thus the Green’s function of our problem is

\[ G(x, y; x_0, y_0) = \sum_{j=1}^{\infty} \frac{\phi_j(x, y) \phi_j^*(x_0, y_0)}{(k^2 - k_j^2)} \quad (3.31) \]

The Eigenfunction Expansion approach will be used intensively for solving the three dimensional Forward Problem for EIT and MEIT later in this thesis.

### 3.4 Discussion

Numerical methods are mainly used in EIT since the images depend very much on the position of electrodes and the shape of the object being imaged. One of the main disadvantages of using analytical solutions is that they can only model simple geometrical shapes. If dynamic images are reconstructed using algorithms such as the Sheffield Filtered Back-projection method, the data are normalised using equation (3.4). As a consequence, the image reconstruction algorithm becomes less sensitive to the position of the electrodes and the shape of the object. Since equation (3.4) implies that the sensitivity matrix is computed once for the uniform case, analytical methods can be used even though they use only simple geometrical shapes. A good example of this was presented by Metherall (1998) where three dimensional images of the human thorax could successfully be reconstructed using an approximate analytical Forward Problem solver.

Whilst numerical methods may be more applicable to complex boundary shapes and for use in iterative image reconstruction algorithms, no compelling in-vivo images using these techniques have been presented to date. All images formed using these methods display considerable artefacts. We therefore maintain that the use of an analytical For-
ward Problem solver at least in the foreseeable future will be useful in EIT, especially when looking at three dimensional image reconstructions. Although there are different methods for solving the Forward Problem analytically, it is important to use the most appropriate one. In this thesis, the Green’s functions technique combined with the Eigenfunction Expansion technique will be used. The reasons for that choice will become clearer in the next chapter.
Chapter 4

Analytical Solution to the Forward Problem for a Finite Right Circular Cylinder

In chapter 1, it was mentioned why the third dimension needs to be addressed in order to reconstruct the images properly. It is also the reason why several groups had begun to address the three dimensional aspects of EIT (Ider et al, 1990), (Pidcock et al, 1995), (Kleinermann et al, 1996), (Kleinermann et al, 1998), (Kleinermann et al, 1998), (Kleinermann et al, 1999), (Kleinermann et al, 1999), (Metherall, 1996), (Morucci et al, 1995) with the aims of:

1. reducing image distortions due to off plane conductivity changes associated with existing two dimensional image reconstructions,

2. increasing the measured boundary datasets, by placing electrodes over the entire surface of the body being imaged, which may translate into increased spatial resolution of the resulting images.

In chapter 1 and chapter 2, it was shown that a prerequisite in all EIT image reconstruction algorithms is access to the solution of the appropriate Forward Problem. In chapter 2, it was briefly mentioned that the repeated solution of the Forward Problem for a uniform conductivity distribution was required for different drives to allow the construction of a sensitivity matrix for the imaged object. Typically, the Forward Problem is solved numerically. Nevertheless, it was stressed in chapter 3 that finding an analytical solution to the Forward Problem is still important and useful. Furthermore, it is possible to reconstruct dynamic images where the Forward Problem has been solved analytically (see chapter 3).
One of the aims of this thesis is to derive a full analytical solution to the Forward Problem. In order to do so, the shape of the object being imaged must be defined. Since the object being imaged is the human thorax, a simple approximate shape could be considered to be a finite right circular cylinder. Therefore, this chapter starts by defining the Forward Problem for which an analytical solution is sought for a finite right circular cylinder. Then, it will introduce an analytical solution for that Forward Problem (called in this thesis “original solution”) and its limitations will be discussed (this will also explain why new forms for the analytical solution to the Forward Problem were necessary in order to overcome some of the limitations associated with this first solution). The third section of this chapter will focus on methods for deriving new forms of the analytical Forward Problem which overcome these limitations. The fourth section will compare the two forms derived in the previous section with the original solution in terms of reconstruction of equipotentials and in terms of potential values computed on five arbitrarily selected points in the three dimensional space. The fifth section presents results of these two forms when different sizes of electrodes are used. It reveals problems of convergence due to the use of small size of electrodes. This behaviour required a study of accelerating techniques to be performed in order firstly to overcome the problems of convergence when small electrodes are used and secondly to accelerate the convergence. The sixth section presents some results for both forms when two acceleration techniques are applied. Finally, the seventh section will validate the analytical solution against other solvers such as the 1/r model, the Boundary Element Method and data collected from a saline tank.

4.1 The Forward Problem

As explained previously, a prerequisite in almost all EIT image reconstruction algorithms is access to the solution of the appropriate Forward Problem. A solution to the Forward Problem allows the potential to be found anywhere in the geometrical object and on
its surface by knowing the conductivity and the injected current. A finite right circular cylinder will be used in this thesis. To allow bipolar delivery of electric current, two rectangular electrodes are attached on its surface. These two rectangular electrodes could be positioned anywhere on its curved surface. Their size does not have to be the same. The geometry of the finite circular cylinder with two electrodes attached on its surface is shown in figure 4.1.

![Figure 4.1. Parameters of the cylinder](image)

In figure 4.1, the position and the size of the two electrodes are defined by:

- $\theta_1$: the angle of electrode 1
- $Z_1$: position of electrode 1 along the z axis
- $W_1$: the width of electrode 1
- $S_1$: the height of electrode 1

For electrode 2, its parameters are given by:

- $\theta_2$: the angle of electrode 2
- $Z_2$: position of electrode 2 along the z axis
- $W_2$: the width of electrode 2
- $S_2$: the height of electrode 2
4.2 Solution to Laplace’s equation using inhomogeneous boundary conditions

In chapter 2, it was shown that the governing equation under quasi-static assumptions was Laplace’s equation. Therefore, the Forward Problem consists of finding the solution to Laplace’s equation for a uniform and isotropic conductivity distribution:

$$\nabla^2 \Phi = 0$$  \hspace{1cm} (4.1)

where

- $\Phi$ is the potential,
- $\nabla^2$ being the Laplacian,

and with inhomogeneous boundary conditions given by

$$\frac{\partial \Phi}{\partial n} = 0 \quad \left\{ \begin{array}{l} \text{on the boundary of the cylinder} \\
\text{which does not include the electrodes,} \\
\text{on only the boundary of the cylinder} \\
\text{which includes the electrodes,} \end{array} \right.$$  \hspace{1cm} (4.2)

where $J$ is the current density, $\sigma$ is the conductivity and $\partial / \partial n$ is the normal derivative to the surface.

A solution to Laplace’s equation is found by solving equation (4.1) expressed in cylindrical co-ordinates as follows:

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

with boundary conditions described above.

4.2.1 Lytle’s solution

The full solution to Laplace’s equation using these boundary conditions is:

$$\Phi(\rho, \phi, z) = - \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{4I_c}{\sigma \pi^3 r^2 n} \left[ \frac{1}{S_1 W_1} \cos\left(\frac{r \pi z_1}{c}\right) \sin\left(\frac{r \pi S_1}{2c}\right) \sin\left(\frac{nW_1}{2a}\right) \cos[n(\phi - \phi_1)] \right] \right\}$$
where
\( I \): current,
\( I_n \): modified Bessel function of first kind of order \( n \),
\( I'_n \): the derivative of the modified Bessel function of first kind of order \( n \).
\( \delta_{0n} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \) (Kronecker symbol)
\( S_1, W_1 \): height and width of the first electrodes
\( S_2, W_2 \): height and width of the second electrodes
\( c \): half-length of the cylinder, \( a \): radius of the cylinder
\( \phi_1, z_1 \): position of the first electrode on the boundary
\( \phi_2, z_2 \): position of the first electrode on the boundary
\( \sigma \): conductivity

This solution was presented by Lytle et al (1979) in his paper entitled “Alternative methods for determining the electrical conductivity of core samples”. In this thesis, Lytle’s solution will be called “original form”.

### 4.2.2 Limitations

Closer inspection of the above expression reveals some important limitations. Firstly, this expression involves only two rectangular electrodes. Secondly, it employs modified Bessel functions which give rise to problems when computing the potential on the boundary of the cylinder. Thirdly, it is a long expression which will give rise to
some problems in its implementation since it is not optimised. Nevertheless, this expression was, to the knowledge of the author, the first published expression for solving Laplace’s equation for a finite right circular cylinder with two rectangular electrodes on its surface and using this expression three dimensional image reconstruction was possible (Kleinermann et al, 1996).

In order to overcome the limitation associated with this formulation, a new form of the Forward Problem was sought which would

1) result in a reduced form,

2) extend the solution to N electrodes

The next section will present an alternative way of solving the three dimensional Forward Problem by solving Poisson’s equation rather than Laplace’s equation.

4.3 Solution to Poisson’s equation using homogeneous boundary conditions

An alternative formulation to the Forward Problem is to consider replacing the electrodes located on the boundary with electrical monopoles located within the cylinder (Smulders and Van Oosterom, 1990) thereby allowing Laplace’s equation to be replaced with Poisson’s equation and thereby transforming the inhomogeneous Neumann boundary conditions (given by equation (4.1)) into homogeneous Neumann boundary conditions.

In our problem, electrodes will be placed arbitrarily on the curved boundary of a circular cylinder injecting a D.C. current. In this situation, equation (4.1) is replaced by Poisson’s equation given by the following form

$$\nabla^2 \Phi = -\frac{\nabla \cdot J(r_0)}{\sigma} \quad (4.4)$$

where $J(r_0)$ is the current at the internal source ($r_0$) and $\sigma$ is the conductivity.
Equation (4.4) can be solved by the application of Green’s functions. For a solution to equation (4.4), the source needs to be replaced by an impulse function. Therefore, equation (4.4) can be rewritten as

\[ \nabla^2 G(r \mid r_0) = -\delta(r - r_0) \]  

(4.5)

where \( G(r \mid r_0) \) is the Green’s function,

\( r_0 \) is the position of the current source,

\( r \) is the position at which we seek a solution.

The Green’s function in equation (4.5) can be obtained by using the Eigenfunctions Expansion technique (Morse and Feshbach, 1953). In this technique, the solution is first expressed as an infinite series of the Eigenfunctions of the required Poisson’s equation. Then substituting back into equation (4.5) and using the orthogonality of the Eigenfunctions, the modal coefficients are determined. Once the modal coefficients are known, the Green’s function can be found (see chapter 3). Having found the Green’s function from equation (4.5), the solution (\( \Phi \)) in (4.4) can be found by multiplying the Green’s function with the current source given in (4.4) and integrating the result of this multiplication over the surface of each electrode. The total solution (\( \Phi \)) at position (\( r \)) is then found by summing the result of each integration over the electrodes, as follows:

\[ \Phi(r) = -\sum_{i=1}^{\text{number of electrodes}} \int_A \int G(r \mid r_0)J_i(r_0) \, dr_0 \]  

(4.6)

where \( J_i \) is the current density applied on the \( i^{th} \) electrode,

\( A \) is the surface area of the \( i^{th} \) electrode.

### 4.3.1 The Green’s function of the first form

The case where two rectangular electrodes are arbitrarily placed on the curved surface of a finite right circular cylinder is considered. The geometry of the finite right circular
cylinder with two electrodes attached is shown in Figure 4.1. Equation (4.5) expressed in cylindrical coordinates is

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial G}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 G}{\partial \phi^2} + \frac{1}{\rho^2} \frac{\partial^2 G}{\partial z^2} = -\frac{1}{\rho} \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(z - z_0) \tag{4.7}
\]

where \( \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(z - z_0) \) are delta functions,

with \( \rho_0, \phi_0, z_0 \) denoting the current source location.

In order to find the modal coefficients, the Eigenfunctions expansion method is applied. In this technique a solution to equation (4.7) is represented by a series of Eigenfunctions of the associated Eigenvalue problem

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial z^2} = 0 \tag{4.8}
\]

with \( V \) satisfying the boundary conditions given by

\( \frac{\partial V}{\partial \phi} \bigg|_{z = z_e} = 0 \) : there is no current flow across the ends of the cylinder,

\( \frac{\partial V}{\partial \rho} \bigg|_{\rho = a} = 0 \) : there is no current flow across the edges of the cylinder (except at the electrodes),

\( V(\phi + 2\pi) = V(\phi) \) : the potential must be periodic.

The even and odd Eigenfunctions of the above equation are

\[
Ve_{n,r,m} = J_n(k_{nm}\rho) \cos(n\phi) \cos(\beta_r(z - c))
\]

\[
Vo_{n,r,m} = J_n(k_{nm}\rho) \sin(n\phi) \cos(\beta_r(z - c)) \tag{4.9}
\]

where \( \beta_r = \frac{\pi}{a} \),

\( J_n \) is the \( n^{th} \) Bessel function of the first kind,

\( k_{nm} \) is the Eigenvalues satisfying the boundary condition,

\( \frac{\partial V}{\partial \rho} \bigg|_{\rho = a} = 0 \),

and called the \( m^{th} \) root of \( J_n'(k_{nm}a) = 0 \).
Once the Eigenfunctions are known, the Green’s function can be expressed as the sum of all the Eigenfunctions. Therefore, the Green’s function for the even case is the sum of all the even Eigenfunctions, given by \( V_{even} \) and is written as

\[
G_e = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} A_{n,m,r} J_n(k_{nm}\rho) \cos(n\phi) \cos(\beta_r(z - c)) + \sum_{r=1}^{\infty} A_{0,0,r} \cos(\beta_r(z - c)).
\]

(4.10)

The second term takes care of the root of the Bessel functions which is equal to zero \( (k_{00} = 0) \). Substituting (4.10) into (4.7), multiplying both sides with \( \rho J_n(k_{nm}\rho) \cos(n\phi) \cos(\beta_r(z - c)) \), using the principle of orthogonality and integrating within the limit of the cylinder, \( A_{n,m,r} \) and \( A_{0,0,r} \) can be found by

\[
(k_{nm}^2 + \beta_r^2)A_{n,m,r} \int_0^a \rho J_n^2(k_{nm}\rho)d\rho \int_0^{2\pi} \cos^2(n\phi)d\phi \int_{-c}^{+c} \cos^2(\beta_r(z - c))dz
\]

\[
= \int_0^a J_n(k_{nm}\rho)\delta(\rho - \rho_0)d\rho \int_0^{2\pi} \cos(n\phi)\delta(\rho - \rho_0)d\phi
\]

\[
\times \int_{-c}^{+c} \cos(\beta_r(z - c))\delta(z - z_0)dz
\]

(4.11)

and

\[
\beta_r^2 A_{0,0,r} \int_{-c}^{+c} \cos^2(\beta_r(z - c))dz = \int_{-c}^{+c} \cos(\beta_r(z - c))\delta(z - z_0)dz
\]

(4.12)

Using the following relationships

\[
\int_0^a \rho J_n^2(k_{nm}\rho)d\rho = \frac{1}{2k_{nm}^2} (k_{nm}^2 - \alpha^2) J_n^2(ak_{nm})
\]

\[
\int_0^{2\pi} \cos^2(n\phi)d\phi = \frac{2\pi}{\sigma_n}
\]
\[ \int_c^{+c} \cos^2(\beta_r(z - c)) = \frac{2c}{\sigma_r} \quad (4.13) \]

where \( \sigma_r = \begin{cases} 1 & r = 0 \\ 2 & r \neq 0 \end{cases} \) and \( \sigma_n = \begin{cases} 1 & n = 0 \\ 2 & n \neq 0. \end{cases} \)

\( A_{n,m,r} \) and \( A_{0,0,r} \) become

\[ A_{n,m,r} = \frac{\sigma_n \sigma_r J_n(k_{nm} \rho_0) k_{nm}^2 \cos(n \phi_0) \cos(\beta_r(z_0 - c))}{2c \pi (a^2 k_{nm}^2 - n^2) J^2_n(k_{nm} a)(k_{nm}^2 + \beta_r^2)} \]

\[ A_{0,0,r} = \frac{\cos(\beta_r(z_0 - c))}{c \pi \beta_r^2 a^2} \quad (4.14) \]

By replacing the coefficients \( A_{n,m,r} \) into equation (4.10), the even Green’s function becomes

\[ G_e(\rho, \phi, z \mid \rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r J_n(k_{nm} \rho_0) J_n(k_{nm} \rho)}{2c \pi (a^2 k_{nm}^2 - n^2) J^2_n(k_{nm} a)} \]
\[ \times \frac{k_{nm}^2 \cos(n \phi) \cos(n \phi_0)}{k_{nm}^2 + \beta_r^2} \]
\[ \times \cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c)) \]
\[ + \sum_{r=1}^{\infty} \frac{\cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c))}{c \pi \beta_r^2 a^2} \quad (4.15) \]

The odd Green’s function is obtained in the same manner and is given by

\[ G_o(\rho, \phi, z \mid \rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r J_n(k_{nm} \rho_0) J_n(k_{nm} \rho)}{2c \pi (a^2 k_{nm}^2 - n^2) J^2_n(k_{nm} a)} \]
\[ \times \frac{k_{nm}^2 \sin(n \phi) \sin(n \phi_0)}{k_{nm}^2 + \beta_r^2} \]
\[ \times \cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c)). \quad (4.16) \]
The complete Green’s function is the sum of the even and odd Green’s functions, given by

\[
G(\rho, \phi, z | \rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r J_n(k_{nm}\rho_0)J_n(k_{nm}\rho)}{2c\pi r^2(a^2k_{nm}^2 - n^2)J_r^2(k_{nm}a)} \times \frac{k_{nm}^2}{k_{nm}^2 + \beta_r^2} \cos(n(\phi - \phi_0)) \\
- \frac{k_{0m}}{\pi r^2 a^2} = \sum_{m=1}^{\infty} \frac{k_{nm}^2J_n(k_{nm}\rho_0)J_n(k_{nm}\rho)}{\pi(k_{nm}^2 + \beta_r^2)(k_{nm}^2 a^2 - n^2)J_r^2(k_{nm}a)}
\]

(4.17)

4.3.2 The second form of the Green’s function

By applying the Mittag-Leffler’s theorem (see appendix A) (Alhargan and Judah, 1991), it can be shown that one of the summations of equation (4.17) can be reduced to a simpler expression, i.e. :

\[
\frac{J_n(i\beta_r\rho) f_n(i\beta_r\rho_0, i\beta_r a)}{4J_n(i\beta_r a)} - \frac{k_{0n}}{\pi r^2 a^2} = \sum_{m=1}^{\infty} \frac{k_{nm}^2J_n(k_{nm}\rho_0)J_n(k_{nm}\rho)}{\pi(k_{nm}^2 + \beta_r^2)(k_{nm}^2 a^2 - n^2)J_r^2(k_{nm}a)}
\]

(4.18)

where \( f_n(\rho, \rho_0) = Y_n(\rho_0)J_n(\rho) - J_n(\rho_0)Y_n(\rho) \).

By replacing the result from equation (4.18) into equation (4.17), a new form of the Green’s function is obtained and is given by

\[
G(\rho, \phi, z | \rho_0, \phi_0, z_0) = -\frac{1}{16c\pi} + \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{\sigma_n J_n(i\beta_r\rho) f_n(i\beta_r\rho_0, i\beta_r a)}{4cJ_n^2(i\beta_r a)} \times \cos(n(\phi - \phi_0)) \cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c)) \\
+ \sum_{n=1}^{\infty} \frac{\cos(n(\phi - \phi_0))}{4cn\pi} \left\{ \left( \frac{\rho}{\rho_0} \right)^n + \left( \frac{\rho}{a} \right)^n \right\}
\]

(4.19)

where \( 0 \leq \rho \leq \rho_0 \leq a \).
For the special case \( \rho_0 = a \), using the Wronskian gives \( f_n(i\beta_r a, i\beta_r a) = \frac{2}{\pi i \beta_r} \) then (4.19) becomes,

\[
G(\rho, \phi, z|a, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{\sigma_n J_n(i\beta_r \rho) \cos(n(\phi - \phi_0)) \cos(\beta_r (z - c))}{2\pi a \alpha r J_n'(i\beta_r a)} \times \cos(\beta_r (z_0 - c)) \]
\[
- \frac{1}{16c^2} + \sum_{n=1}^{\infty} \left( \frac{2}{\pi} \right)^n \frac{\cos(n(\phi - \phi_0))}{2\pi nc} \]

where \( 0 \leq \rho \leq a \).

### 4.3.3 The potential using the first form of the Green’s function

The potential \( \Phi_1 \) due to the first electrode is found by replacing \( G(\rho, \phi, z|\rho_0, \phi_0, z_0) \) from equation (4.17) into equation (4.6) and integrating over the surface of the rectangular electrode (see appendix B) which has a length \( S_1 \) and a width \( W_1 \) with \( \Delta_1 \) being the subtended angle determined as \( \Delta_1 = \tan^{-1}(\frac{W_1}{2S_1}) \) is given by

\[
\Phi_1(\rho, \phi, z) = -\int_{\phi_1 - \Delta_1}^{\phi_1 + \Delta_1} J_1(\rho_0, \phi_0, z_0)G(\rho, \phi, z|\rho_0, \phi_0, z_0) d\phi_0 dz_0. \quad (4.21)
\]

By substituting equation (4.17) into equation (4.21) and integrating over the electrode’s surface.

\[
\Phi_1(\rho, \phi, z) = -\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r 2a J_1 J_n(k_{nm}\rho) \Delta_1}{c\pi \beta_r (a^2 k_{nm}^2 - n^2) J_n(k_{nm} a)} \times \frac{k_{nm}^2}{k_{nm}^2 + \beta_r^2} \cos(n(\phi - \phi_1)) \text{sinc}(n\Delta_1) \]
\[
\times \cos(\beta_r (z - c)) \cos(\beta_r(z_1 - c)) \sin(\beta_r \frac{S_1}{2}) \]
\[
- \sum_{r=1}^{\infty} 4J_1 \Delta_1 \cos(\beta_r (z - c)) \cos(\beta_r(z_1 - c)) \sin(\beta_r \frac{S_1}{2}) \]

\[= -\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r 2a J_1 J_n(k_{nm}\rho) \Delta_1}{c\pi \beta_r (a^2 k_{nm}^2 - n^2) J_n(k_{nm} a)} \times \frac{k_{nm}^2}{k_{nm}^2 + \beta_r^2} \cos(n(\phi - \phi_1)) \sin(n\Delta_1) \]
\[
\times \cos(\beta_r (z - c)) \cos(\beta_r(z_1 - c)) \sin(\beta_r \frac{S_1}{2}) \]
\[
- \sum_{r=1}^{\infty} 4J_1 \Delta_1 \cos(\beta_r (z - c)) \cos(\beta_r(z_1 - c)) \sin(\beta_r \frac{S_1}{2}) \]

\[= -\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r 2a J_1 J_n(k_{nm}\rho) \Delta_1}{c\pi \beta_r (a^2 k_{nm}^2 - n^2) J_n(k_{nm} a)} \times \frac{k_{nm}^2}{k_{nm}^2 + \beta_r^2} \cos(n(\phi - \phi_1)) \sin(n\Delta_1) \]
\[
\times \cos(\beta_r (z - c)) \cos(\beta_r(z_1 - c)) \sin(\beta_r \frac{S_1}{2}) \]
\[
- \sum_{r=1}^{\infty} 4J_1 \Delta_1 \cos(\beta_r (z - c)) \cos(\beta_r(z_1 - c)) \sin(\beta_r \frac{S_1}{2}) \]
where \( \text{sinc}(n \Delta_1) = \frac{\sin(n \Delta_1)}{n \Delta_1} \)

and \( J_1 = \frac{I}{\sigma S_1 W_1} \), with \( I \) being the current injected and \( \sigma \) being the conductivity.

The same method is then applied to the electrode(s), with the desired total potentials being the sum of all the potentials due to the electrodes. In this chapter, the above equation is called “first form”.

### 4.3.4 The potential using the second form of the Green’s function

By substituting the second form of the Green’s function (4.20) into equation (4.21) and integrating over the area of the first electrode, a second analytical expression can be derived.

\[
\Phi_1 (\rho, \phi, z) = \frac{2J_1 \Delta_1 S_1 a}{16 \pi} - \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{2J_1 \sigma_n J_n(i \beta_r \rho) \cos(n(\phi - \phi_1)) \sin(n \Delta_1)}{n \rho \pi \beta_r^2 J_r(i \beta_r a)} \times \cos(\beta_r (z - c)) \cos(\beta_r (z_1 - c)) \sin(\beta_r S_1 / 2) \]

\[
- \sum_{n=1}^{\infty} \left( \frac{\rho}{a} \right)^n \frac{a S_1 J_1 \cos(n(\phi - \phi_1)) \sin(n \Delta_1)}{\pi n^2 c}
\]

\[
(4.23)
\]

where \( 0 \leq \rho \leq a \).

By using the well known relationships between Bessel functions and Modified Bessel functions (Abramowitz and Stegun, 1965),

\[
J_n(i x) = i^n I_n(x)
\]

\[
J'_n(i x) = i^{n-1} I'_n(x)
\]

and by replacing these relations in equation (4.23), the second solution then becomes:
\[
\Phi_1(\rho, \phi, z) = \frac{2J_1 \Delta_1 S_1 a}{16c\pi} - \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{2J_1 \sigma_n I_n(\beta_r \rho) \cos(n(\phi - \phi_1)) \sin(n\Delta_1)}{nc\pi \beta_r^2 I_n(\beta_r a)} \\
\times \cos(\beta_r(z - c)) \cos(\beta_r(z_1 - c)) \sin(\beta_r S_1/2) \\
- \frac{aS_1 J_1}{\pi c} \sum_{n=1}^{\infty} \left( \frac{\rho}{a} \right)^n \cos(n(\phi - \phi_1)) \sin(n\Delta_1) \tag{4.24}
\]

where \( \beta_r = \frac{r\pi}{2c} \).

Again, the total potential is found by summing the potentials due to all the electrodes\(^8\).

In this chapter, the above equation is called “second form”.

Table 4.1. Summary of the two forms for a finite right circular cylinder

---

\(^8\) More details on the derivation of the second form can be found in appendix B.
4.4 Comparison between Poisson’s solution using homogeneous Neumann’s boundary conditions and Laplace’s solution using inhomogeneous Neumann’s boundary conditions

The reduced form given by equation (4.24) is compared with the original form (4.3) in terms of reconstruction of the equipotentials on the boundary and within the x-y plane at z=0 and in terms of potential calculated on five points situated both on the boundary of and inside the cylinder but all in the same plane (z=0). In the light of Lytle’s results (Lytle et al., 1979), the cylinder was chosen to have a height of 2.7cm and a radius of 0.715cm. The conductivity was uniform (1 Scm⁻¹). The two rectangular electrodes had a height of 0.26cm and a width of 0.14cm. The applied current was ±27.847 mA. The first electrode configuration is the “polar configuration”. In that configuration, the first electrode is positioned at an angle of 90 degrees and the second electrode is positioned at an angle of -90 degrees. Both electrodes in that configuration are situated in the same plane z=0 cm. To demonstrate the generality of the analytical configuration, an “asymmetric configuration” was chosen for the second configuration. In that configuration, the electrodes are located in different z planes. The first electrode is positioned at 35 degree at z=0 cm. The second electrode is positioned at 270 degree at z=1 cm.

4.4.1 Reconstruction of the equipotentials

To reconstruct the equipotentials, the potential on each node of a mesh (having 187 points inside at plane z=0) using analytical expressions (4.24) and the original form (4.3) was computed and displayed (using the routines from University of Nijmegen’s Boundary Element Method Software (Van Oosterom)). The total summation for the two series involved in the analytical expressions (4.24) and in the original form (4.3) were fixed to 150 terms since after 150 terms, the ratio involving the modified Bessel
functions \( \frac{I_n(\beta, \rho)}{L_0(\beta, \alpha)} \) reaches the machine precision (32 bits) or in the representation of floating numbers is \((+1.7E+308, -1.7E-308)\).

For both expressions, the reconstructed equipotentials have the same shape. Therefore, only the reconstructed equipotentials for equation (4.24) are presented in this section.

Firstly, the reconstruction of the equipotentials are presented for the polar configuration. Figure 4.2 shows the reconstructed equipotentials along the z axis on the boundary of the cylinder. Figure 4.3 shows the reconstructed equipotentials in the xy plane at z=0.

Secondly, the reconstructed equipotentials for the asymmetric configuration are presented. Figure 4.4 shows the reconstructed equipotentials along the z axis on the bound-
ary of the cylinder. Figure 4.5 shows the reconstructed equipotentials in the xy plane at z=0 for electrode 1.

![Figure 4.5](image-url)

**Figure 4.4.** Reconstruction of the equipotentials for the asymmetric drive configuration on the boundary.

![Figure 4.5](image-url)

**Figure 4.5.** Reconstruction of the equipotentials for the asymmetric drive configuration in the xy plane where one of the electrodes is located (electrode 1).

### 4.4.2 Potentials on five arbitrarily selected points

Table 1 shows the results for five points for the original form (4.3) and the reduced form given by equation (4.24) using the polar configuration. Similar correspondence between calculated potentials using the two expressions was obtained for other points including those not in the plane z=0.
Table 4.2. Comparison of potentials calculated for five points situated both on the original boundary and inside of the cylinder but in the same plane ($z=0$) for using the reduced form (equation $(4.24)$) using the polar configuration (mV).

Table 4.3 shows the results for these five points for the original form presented in $(4.3)$ and the reduced form given by equation $(4.24)$ using the asymmetric configuration. Similar correspondence in calculated potentials was obtained for other points not in the plane ($z=0$) plane.

<table>
<thead>
<tr>
<th>Point</th>
<th>Original Form</th>
<th>Reduced Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0 0.715</td>
<td>-80.5595219</td>
<td>-80.5595217</td>
</tr>
<tr>
<td>-0.1396 0.7013</td>
<td>-30.1396858</td>
<td>-30.1396857</td>
</tr>
<tr>
<td>-0.2736 0.6606</td>
<td>-15.1608233</td>
<td>-15.1608223</td>
</tr>
<tr>
<td>-0.3972 0.5945</td>
<td>-9.3950317</td>
<td>-9.3950316</td>
</tr>
<tr>
<td>-0.5056 0.5056</td>
<td>-6.2164471</td>
<td>-6.2164470</td>
</tr>
</tbody>
</table>

Table 4.3. Comparison of potentials calculated for five points situated both on the original boundary and inside of the cylinder but in the same plane ($z=0$) for using the reduced form (equation $(4.24)$) using the asymmetric configuration (mV).
4.4.3 Discussion

Figures 4.2 through to figure 4.5 show that the reconstructed equipotentials are well behaved and have the expected shape. It must be pointed out that the “distortion” around the electrodes is due to the mesh employed and is also due to the problems associated with the ‘0’ volt equipotential and is an artefact of the mesh/plotting software. Furthermore, table 1 and table 2 show that the reduced form has the same potential values as the original form up to seven significant figures. After seven significant figures the result from the two expressions start to differ. This is due to small differences starting to appear in the different computational implementations of the two expressions. It is also interesting to note that equation (4.22) can be reduced to equation (4.24) by applying the Mittag-Leffler’s theorem (Alhargan and Judah, 1991) when identical electrodes used in bipolar drive configurations are employed. In other words, Mittag-Leffler’s theorem transforms equation (4.22) into equation (4.24) which is a reduced form of the solution given in (Lytle et al, 1979). The above is an interesting result and presents some justification for employing simpler electrode models for EIT applications, as already employed by some (Avis, 1993), (Smulders and Van Oosterom, 1990). However, some caveats to the above findings should be noted. Firstly, the relationship between Laplace’s equation and Poisson’s equation exists only because the two rectangular electrodes have the same size and inject a direct current. Secondly, this relationship exists because the source in the Poisson’s equation is very close to the boundary. Further work is required to ascertain if the equivalence between Laplace’s and Poisson’s results can be generalised to different sized electrodes, non-bipolar drive configurations and different geometrical bodies.

Finally, the Green’s function technique as used here in solving Poisson’s equation using homogeneous boundary conditions for a finite right circular cylinder presents four advantages over the Mode Matching technique used in (Lytle et al, 1979) to solve Laplace’s equation. Firstly, it provides two forms of the solution to Poisson’s equation given by equation (4.22) and (4.24). Secondly, it provides directly the reduced form given by equation (4.24) of the solution given in (Lytle et al, 1979). Thirdly, the first form given by equation (4.22) uses Bessel functions which are more stable than the modified Bessel
functions used in equation (4.24) and in (Lytle et al, 1979) and therefore affords a better computational implementation. Fourthly, both forms can handle n-number of electrodes of different sizes and polarity.

4.5 Use of different sized electrodes

This section presents results for both forms and explores their limitations when small electrodes are used. The analytical solution given in (4.22) was applied to a homogeneous finite right circular cylinder with a radius of 0.715 cm and a height of 2.7 cm. Two square electrodes were placed on the curved boundary of the cylinder. A current of 0.002 mA was applied to the cylinder. The electrical conductivity was isotropic and uniform (1.0 S/cm). Numerical results are presented for two types of electrodes; “large electrodes” and “small electrodes”. The large electrodes had a height and width of 0.14 cm, while the small electrodes had a height and width of 0.035 cm. For each type of electrode, two configurations of the electrodes on the boundary of the circular cylinder were considered. The first electrode configuration known as the “adjacent configuration” is shown in figure 4.6. In this configuration, the two electrodes are positioned in the same z plane. The first electrode is positioned at an angle of $-5.675$ degrees and the second electrode is positioned at an angle of $+5.675$ degrees. This configuration has been chosen since it provides the largest range of potentials. To demonstrate the generality of the analytical solution, an “asymmetric configuration” was chosen for the second configuration as shown in figure 4.6. The electrodes are located in different z planes. The first electrodes is located at the plane $z=1.0$ cm and is positioned at the angle of 200 degrees in that plane. The second electrode is located at the plane $z=0.0$ cm and is positioned at an angle of 35 degrees in that plane. To demonstrate the first form of the analytical solver, reconstruction of the equipotentials will be shown in the next sections for both sizes of the electrodes and for both drive configurations.
Numerical results of the first form

To reconstruct the equipotentials, the potential on each node of a mesh (having 418 points on the surface and 187 points inside at plane z=0) using the analytical expression given
in (4.22) was computed and displayed. The total summation for the first two series was fixed to 90 terms since it guarantees that the roots of the Bessel functions are accurate. The summation for the inner series was fixed at 2018 terms for small electrodes and 518 terms for large electrodes. The difference in the number of terms used is explained by the fact that the sine function contains the height of the electrodes as an argument and requires less terms for large electrodes than for small electrodes to ensure that a complete cycle of sine function is included in the summation.

Figure 4.7 and figure 4.8 show the resulting equipotentials along the z axis and also in the plane of the electrodes for the adjacent configuration and the asymmetric configuration using large electrodes. In both cases the equipotentials are quite smooth and have the expected shape as illustrated by the x-y plane plots where the equipotentials intersect with the boundary of the cylinder at right angles, consistent with the imposed boundary conditions.

Figure 4.9 and figure 4.10 show the reconstructed equipotentials for the adjacent configuration and the asymmetric configuration using small electrodes. In both cases the reconstructed equipotentials are distorted from the expected form and some artefacts appear.

Figure 4.7. Reconstruction of the equipotentials for the adjacent configuration using large electrodes.
Figure 4.8. Reconstruction of the equipotentials for the asymmetric configuration using large electrodes.

Figure 4.9. Reconstruction of the equipotentials for the adjacent configuration using small electrodes.
4.5.2 Numerical results of the second form

In order to compare this second form of Poisson’s solution with the first form of Poisson’s solution, the previous tests were applied. The total summation for both series were fixed to 150 terms since after 150 terms the ratio involving the modified Bessel functions reaches machine precision (32 bits).

Figure 4.11 and figure 4.12 show the resulting equipotentials along the z axis and also in the plane of the electrodes for the adjacent configuration and the asymmetric configuration using large electrodes. In both cases the equipotentials are quite smooth and have the expected shape since they intersect with the boundary of the cylinder at right angles. Figure 4.13 and figure 4.14 show the reconstructed equipotentials for the adjacent configuration and the asymmetric configuration using small electrodes. In both cases the reconstructed equipotentials are again distorted from the expected form and some artefacts appear.
Figure 4.11. Reconstruction of the equipotentials for the adjacent configuration using large electrodes.

Figure 4.12. Reconstruction of the equipotentials for the asymmetric configuration using large electrodes.
When larger electrodes are employed the alternative form of the analytical expression for the solution of Poisson’s equation behaves well and appears to give good results on the basis of the resultant equipotentials (see figure 4.11 and figure 4.12). However, when small electrodes are used, the equipotentials are distorted for both electrode configurations (see figure 4.13 and figure 4.14). Whilst not perfect the second form does, however, appear to be better behaved than the first form.
4.5.3 Discussion

From the above studies performed on both forms, it can be seen that both forms reconstruct the equipotentials along the z axis and inside the object correctly. However, both forms start to have convergence problems when small electrodes are used since the reconstructed equipotentials appear to be distorted. Since the accuracy of the sensitivity matrix is dependent on the accurate computation of the gradient, the results of the above studies in terms of the shape and form of the resulting equipotentials would appear to indicate that for large electrode configurations an accurate sensitivity matrix could be calculated. However, the resulting form of the equipotentials for small electrodes leads to errors being introduced in the sensitivity matrix and therefore possible distortions in the reconstructed EIT images. This also implies that both the first and second form of the analytical solver are converging very slowly as the size of electrodes decreases.

There is a slight difference between the first form of Poisson’s solution and the second form of Poisson’s solution in terms of computational time since it takes about 15 seconds for the first form to compute the potential on one point (using a SiliconGraphics(O2) MIPS R5000 SC, 180 MHz processor with 90 terms used for the first two series and 518 terms for the inner series) and it takes about 10 seconds for the second form of Poisson’s solution to compute the potential on one point (using 150 terms for the two series).

One direct method for improving the convergence would be to increase the number of terms. Although it should improve the convergence, three problems remain:

1. The accuracy of the computation of the roots for the Bessel functions in the first form are questionable for a large number of terms and these inaccuracies may seriously impact the accuracy of the solution.

2. The second form is limited to a certain number of terms since the ratio \( \frac{J_n(\beta \rho)}{J_{n-1}(\beta \rho)} \) reaches the machine precision very quickly.

3. The increase in the number of terms results also in an increase of the computational time for both forms which obviously will make the computation of the full three dimensional sensitivity matrix very long.

The problem of convergence for series is a well known problem and some acceleration techniques have been developed to overcome this problem. In order to improve the con-
vergence behaviour of both forms for small electrodes and to try to reduce the computational time, the application of accelerating convergence techniques would appear worth considering. The next section presents some results when two accelerating convergence techniques are applied.

### 4.6 Study of convergence for both forms when two accelerating techniques are applied

In this section, two acceleration techniques (namely Euler’s transformation and Epsilon’s transformation) will be applied to both forms. Euler’s transformation technique was chosen since it performs well in the convergence of alternating series. Full details of the Euler’s algorithm can be found in (Press et al, 1982). The Epsilon transformation was chosen since it is firstly a generalization of the Aitken’s process (Aitken, 1926), secondly it is a non linear transformation from sequences to sequences and finally, it is directly related to the Shank’s transformation (Shank, 1955). The Epsilon transformation algorithm used in this paper was implemented from Brezinski’s book (Brezinski, 1978). In order to investigate the performance of each technique, two studies are presented in this section.

The first study consists of reconstructing the equipotentials using both forms of analytical expression with the use of small electrodes when these two accelerating techniques are applied and visual inspection of the equipotentials to ascertain if the distortions have disappeared.

The second study consists of analysing the convergence rate of both forms on three arbitrarily selected points situated on the boundary and inside the cylinder (figure 4.15).
For each point and for each form, a graph is used for displaying the rate of the convergence for that point. Each graph shows three plots. The first plot corresponds to the summation of the analytical expressions when no acceleration convergence techniques were applied and is termed “direct summation”. The second plot corresponds to the summation result when Euler’s transformation is applied. The third plot corresponds to the summation of the series when Epsilon’s transformation is applied. Furthermore, the graphs also show the relationship between three different parameters (namely the number of significant digits, the difference between the result of the algorithm and that summing the series to a certain maximum (or total summation), and the number of terms required to achieve a certain precision (significant digits)).

Both studies are performed on the same parameters of the cylinder used in the previous section and for the same configuration of electrodes in order to see directly if improvements are evident. The next section presents the results for the reconstruction of the equipotentials.
4.6.1 Reconstruction of the Equipotentials

Figure (4.16) and figure (4.17) show that the equipotentials have the expected shape for the adjacent configurations and for both forms when Euler’s transformation is used.

Figure 4.16. Reconstruction of the equipotentials for the adjacent configuration using small electrodes when Euler’s transformation is applied to the first form.

Figure 4.17. Reconstruction of the equipotentials for the adjacent configuration using small electrodes when Euler’s transformation is applied to the second form.

Figure 4.18 and figure 4.19 show also that the equipotentials have the expected shape for the asymmetric configurations and for both forms when Euler’s transformation is used.
Figure 4.18. Reconstruction of the equipotentials for the asymmetric configuration using small electrodes when Euler’s transformation is applied to the first form.

Figure 4.19. Reconstruction of the equipotentials for the asymmetric configuration using small electrodes when Euler’s transformation is applied to the second form.

However, small artefacts appear in the plane of the equipotentials arising from for the adjacent drive configuration when the Epsilon’s transformation is applied (see figure 4.20). Nevertheless, the equipotentials have the expected shape for the adjacent configurations when Epsilon’s transformation is applied to the second form (see figure 4.21).
Figure 4.20. Reconstruction of the equipotentials for the adjacent configuration using small electrodes when Epsilon’s transformation is applied to the first form.

When Epsilon’s transformation is applied to both forms using the asymmetric configuration, the equipotentials have the expected shape.
Figure 4.22. Reconstruction of the equipotentials for the asymmetric configuration using small electrodes when Epsilon’s transformation is applied to the first form.

Figure 4.23. Reconstruction of the equipotentials for the asymmetric configuration using small electrodes when Epsilon’s transformation is applied to the second form.
4.6.2 Convergence behaviour

The second study is the convergence study on three arbitrarily selected points. Figure 4.24 shows the graph for the adjacent drive configuration and for the first point when small electrodes are used. It can be seen that the plot of the direct summation shows that the first form has not converged since no significant figure is reached after 300 terms. However, the plots related to both acceleration convergence techniques show that the convergence behaviour of the first point using small electrodes has improved since five significant figures are reached after 180 terms for Euler’s transformation and 200 terms for Epsilon’s transformation. Furthermore, the relative error for both acceleration techniques decreases with the number of significant figures. The relative error is smaller for Epsilon’s transformation than for Euler’s transformation. However, Epsilon’s transformation requires more terms than Euler’s transformation to reach five significant figures. The study of the convergence on that point suggests that the convergence improves by applying the accelerating convergence techniques to the first form.

![Figure 4.24. Convergence of the point 1 for the adjacent configuration using small electrodes with the first form.](image)

Direct sum points are not shown since using this approach results in a lack of convergence.
When the same study is performed using the second form (see figure 4.25), there are no significant figures obtained for the direct sum. Even employing Euler’s transformation and Epsilon’s transformation, only one significant figure is obtained after 8 terms. This shows that the convergence is very slow even when the acceleration techniques are applied.

![Graph showing convergence of the point 1 for the adjacent configuration using small electrodes with the second form.](image)

Figure 4.25. Convergence of the point 1 for the adjacent configuration using small electrodes with the second form.

Figure 4.26 shows the graph for the second point and for the first form when small electrodes are used using the adjacent configuration. The convergence is slow even when accelerating techniques are used since Epsilon’s transformation and Euler’s transformation give the same number of significant figures (i.e., two significant figures) as the Direct sum. However, Euler’s transformation converges faster than Epsilon’s transformation and the Direct sum. When both accelerating techniques are applied, the relative error decreases faster than with the Direct method.
Figure 4.26. Convergence of the point 2 for the adjacent configuration using small electrodes with the first form.

Figure 4.27 (see next page) shows the graph of the convergence behaviour for the second point for the adjacent drive configuration using the second form. It can be seen that convergence is reached by using the Direct summation, Euler’s transformation or Epsilon’s transformation since fourteen significant figures are reached in a few terms. Furthermore, the relative error is decreasing with the number of terms and with the number of significant figures reached.

Figure 4.28 (see next page) shows the graph for the third point when small electrodes are used with the adjacent configuration and with the first form. The convergence behaviour overall is better since the Direct sum gives five significant figures and less number of terms need to be recruited than Euler’s transformation. However, the relative error seems to be better for Euler’s transformation than the Direct sum.
Figure 4.27. Convergence of the point 2 for the adjacent configuration using small electrodes with the second form.

Figure 4.28. Convergence of the point 3 for the adjacent configuration using small electrodes with the first form.
It must be stressed that the graph for the second form using the adjacent configuration is not shown in this thesis since there is no convergence at all for the adjacent configuration using either Euler’s transformation, Epsilon’s transformation or the Direct summation. This shows that the convergence is still one of the remaining problems associated with the derived analytical expressions.

For the asymmetric configuration and using the first form, the convergence of the first point to a solution is starting using the Direct summation method although only one significant figure is obtained after 210 terms (see figure 4.29). When Euler’s transformation and Epsilon’s transformation are applied, the convergence improves since four significant figures are reached after 225 terms and the relative error is decreasing according to the number of significant figures reached (see figure 4.29). The relative error for both accelerating convergence techniques is of the same order.

Figure 4.29. Convergence of the point 1 for the asymmetric configuration using small electrodes with the first form.
The convergence study of the same point for the asymmetric configuration using the second form shows also that the convergence is very slow since only one significant figure is reached for the Direct sum, Euler’s transformation and for the Epsilon’s transformation (see figure 4.30). Euler’s transformation reaches one significant figure after 110 terms, while Epsilon’s transformation reaches one significant figure after 120 terms. The Direct sum reaches one significant figure after 130 terms. In terms of the relative error, it can be seen that the relative error is smaller for Epsilon’s transformation than for the Euler’s transformation and the Direct sum.

![Figure 4.30. Convergence of the point 1 for the asymmetric configuration using small electrodes with the second form.](image)

Figure 4.31 shows the graph for the second point when small electrodes are employed using the asymmetric configuration with the first form. It can be seen that the convergence is very slow even with the application of accelerating convergence techniques. Indeed, only two significant figures are obtained for the Direct sum and for Euler’s transformation after 45 terms. However, Epsilon’s transformation seems to be better since two significant figures are reached after 25 terms.

For all the methods, the relative error decreases with the number of terms and it can be said that the relative error is the same order for the Direct sum, Euler’s transformation
and Epsilon’s transformation. However, the number of terms for reaching that order is not the same (see figure 4.31).

Figure 4.31. Convergence of the point 2 for the asymmetric configuration using small electrodes with the first form.

Figure 4.32 (see next page) shows the graph of the convergence behaviour for the second point using the asymmetric drive configuration with the second form. It can be seen that convergence is reached using all methods since again fourteen significant figures are reached in only a few terms. Furthermore, the relative error is decreases with the number of terms and with the number of significant figures reached.

Figure 4.33 (see next page) shows the graph for the third point when small electrodes are used with the asymmetric configuration using the first form. It can be seen that the convergence of the third point with the Direct summation is slow since only one significant figure is obtained. When using Euler’s and Epsilon’s transformations, the convergence of the third point is faster since four significant figures are reached (see figure 4.33). Furthermore, the convergence of the third point is not too slow for both acceleration convergence methods. The relative error also decreases with the number
of terms. The order of the relative error is the same for both accelerating convergence techniques when four significant figures are reached.

Figure 4.32. Convergence of the point 2 for the asymmetric configuration using small electrodes with the second form.

Figure 4.33. Convergence of the point 3 for the asymmetric configuration using small electrodes with the first form.
Figure 4.34 shows the graph for the convergence of the third point using the asymmetric configuration with the second form. It can be seen that the accelerating convergence techniques improve the convergence since four significant figures are obtained compared to one significant figure obtained when using the direct summation method. The relative error also decreases with the number of terms and is smaller with Epsilon’s transformation compared with the Direct sum or Euler’s transformation when four significant figures are obtained.

![Graph showing convergence of the third point](image1)

**Figure 4.34.** Convergence of the point 3 for the asymmetric configuration using small electrodes with the second form.

### 4.6.3 Discussion

To summarize the above results for both forms in terms of convergence when two accelerating techniques are applied to them and for two configurations (namely adjacent configuration and asymmetric configuration), the following points can be deduced:

1) All the artifacts which appear when small electrodes are used, are significantly reduced when both accelerating techniques are applied to both forms and for both configurations.
2) The behaviour of the convergence on three points for both forms and for both configurations in terms of improvement when two accelerating techniques are applied to them can be summarized by the following tables:

<table>
<thead>
<tr>
<th>Convergence behaviour on</th>
<th>Direct sum for the first form</th>
<th>Direct sum for the second form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point 1</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Point 2</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Point 3</td>
<td>~</td>
<td>--</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Convergence behaviour on</th>
<th>Euler's method for the first form</th>
<th>Euler's method for the second form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point 1</td>
<td>~</td>
<td>--</td>
</tr>
<tr>
<td>Point 2</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Point 3</td>
<td>~</td>
<td>--</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Convergence behaviour on</th>
<th>Epsilon's method for the first form</th>
<th>Epsilon's method for the second form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point 1</td>
<td>~</td>
<td>--</td>
</tr>
<tr>
<td>Point 2</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Point 3</td>
<td>~</td>
<td>--</td>
</tr>
</tbody>
</table>

Table 4.4 Summary of the convergence behaviour for the adjacent configuration on three selected points where + : good (more than five significant figures), ~: average (more than three significant figures), - : poor (more than one significant figures) and -- : no good (zero or one significant figures)
Table 4.5 Summary of the convergence behaviour for the asymmetric configuration on three selected points where + : good (more than five significant figures), ~: average (more than three significant figures), - : poor (more than one significant figures) and -- : no good (zero or one significant figures)

From this study, it can be concluded that the Direct sum for the adjacent configuration performs better for the first form than the second form for point 1 and point 3 when accelerating techniques are applied. Nevertheless, the second form performs better than the first form for point 2, but the convergence of point 1 and point 3 for the second form stays very poor even though accelerating convergence techniques have been applied to it.

For the asymmetric configuration, it can also be seen that the convergence for the first form is poor for the three points with no accelerating techniques applied, while for the second form, the convergence is poor for point 1 and point 3 and remains very good for point 2. When accelerating techniques are applied, the convergence for the first form improves for point 1 and point 3 and does not improve for point 2. For the second form, the convergence is excellent for point 2 and improves for point 3. However, it does not improve for point 1.
Therefore, it can be concluded that the trend of the convergence between both forms changed according to the position of the points. The trend is in general improved by applying accelerating convergence techniques. It can be seen that the convergence of the first form of the solution is well behaved for points on the boundary and very slow for points inside the plane. Conversely, with the second form of the solution, the convergence is very good for points inside the plane and is poor for points on the boundary. This is due to the ratio involving the modified Bessel’s function in the second form which behaves very differently for points inside the domain and for points on the boundary. For points inside, this ratio behaves as a step function making the second form converge very quickly (see figure 4.35), while for points on the boundary the ratio behaves like a function converging to one (see figure 4.36) and since no other terms in the series can rapidly influence the convergence of the second form, the second form of the solution converges very slowly for the points on the boundary. This suggests the use of a hybrid approach, using both forms of the solution for different portions of the imaged object as appropriate. This hybrid approach would use the first form of the solution for points on the boundary and the second form of the solution would be used for points inside the plane. Furthermore, Euler’s transformation will be chosen rather than Epsilon’s transformation since it has been shown to be the best accelerating convergence technique for the two forms of the solution.

Figure 4.35. Behaviour of the ratio involving the modified Bessel functions in the second form for a point situated inside a plane and at half the radius of the cylinder.
4.7 Voltage model validation

As the second form is derived from the first form and as the second form is not very good for points located on the boundary (see previous section), only the first form will be compared with other solvers in this section. The comparison is done by comparing the boundary voltage profile (BVP) of the first form with the boundary voltage profile from the 1/r model (Metherall, 1996), University of Nijmegen’s Boundary Element Method (BEM) software and a boundary voltage profile from a saline filled cylindrical tank. Two studies have been performed. The first study shows the comparison between the BVP from the first form with the BVP from the 1/r model and from the BEM. The second study compares the BVP from first form with the BVP of the 1/r model and a saline tank. Both studies are performed with 32 electrodes placed in an interleaved configuration.

4.7.1 First study

The finite right circular cylinder has a radius of 1.0 cm and a height of 2.0 cm. A current of 0.002 mA was applied to the cylinder. The 32 electrodes were placed close to the top of the cylinder a z=1.75cm (see figure 4.37). The square electrodes had a height of
0.06cm and a width of 0.06cm. The electrical conductivity was isotropic and uniform (1.0 $S\text{cm}^{-1}$).

Figure 4.37. Electrodes placed close to the top of the cylinder in an interleaved configuration.

The comparison between the first form with the $1/r$ model and the BEM is shown in figure 4.38.
The inspection of figure 4.38 shows that the shape of the boundary voltage profiles for numerical method (BEM) and the analytical method are very similar, but the boundary voltage profiles of the 1/r model are different for the receive electrodes close to the drive electrodes.

4.7.2 Second study

The second study was performed by Metherall (1998) with the use of the first form. The results shown in this section were taken from Metherall’s thesis (Metherall, 1998). His study is important since it compares the boundary data generated from the analytical solution (first form), from the 1/r model and from real data collected from four electrode planes of a saline filled tank. There were four electrode planes each with 8 receive pairs in an interleaved configuration. The first plane of electrodes was situated at $z=0.25$ cm, the second plane was at $z=0.75$, the third plane at $z=1.25$ and the fourth plane at $z=1.75$. The profiles correspond to a single drive pair and thus only a selection of the boundary voltages eminating from the homogeneous cylindrical conductivity distribu-
tion were presented. Both computed data sets are based on dimensions of the physical electrode planes (diameter 235mm, plane separation 60mm), but the analytical solution also takes account of the overall height of the saline volume (height 420mm, plane 1 at 120mm). Furthermore, Metherall normalised each profile to its maximum value. Figure 4.39 shows his results (Metherall, 1998) reproduced here with permission.
Metherall (1998) reached the conclusion that the results given by the analytical method were consistent with the tank and the 1/r model. Furthermore, the fourth plane did indicate that some differences started to appear between the 1/r model and the analytical solution with the saline tank. This was due to the fact that the 1/r model assumes an infinite half space. This suggests strongly that the analytical solution is more accurate and should be employed in future studies.

Figure 4.39. Normalised boundary voltage profiles for 8 receive pairs over the 4 electrode planes where “+” is analytical solution, “x” is the 1/r model, and “o” is the tank.
4.8 Discussion

This chapter has presented two methods for deriving the solution to the Forward Problem for a finite right circular cylinder on the curved surface of which two rectangular electrodes are placed arbitrarily for injecting a direct current. The first method given by equation (4.3) was derived by solving Laplace’s equation with inhomogeneous Neumann’s boundary conditions using the Mode Matching method. However, this solution presented five disadvantages namely:

1. It involves a long derivation for obtaining the solution given by equation (4.3),

2. Convergence is not easily obtained for small electrodes and for points located on the boundary of the circular cylinder,

3. It is limited to a certain number of terms since the modified Bessel functions terms reach quickly the numerical limit of the machine,

4. Although it involves only a double series, the speed of computation is slow.

5. It is difficult to generalize to n-electrodes.

In an attempt to overcome the above limitations an alternative form of the analytical expression was sought. Instead of solving Laplace’s equation with inhomogeneous Neumann’s boundary conditions, it was thought that solving Poisson’s equation with homogeneous Neumann’s boundary conditions would present some advantages since different forms of the solution to the Forward Problem may be obtained. Furthermore, the derivation of the solution can be accelerated by the use of the Green’s function with the Eigenfunction Expansion technique. By solving Poisson’s equation with homogeneous Neumann’s boundary conditions, a first solution was derived. Due to the flexibility offered by the Green’s function combined with the Eigenfunction Expansion technique, a second form could be derived using the Mittag-Leffler’s theorem (Alhargan and Judah, 1991). Analysis presented in this chapter shows that the second form is identical to the first solution given by Lytle, if both drive electrodes are identical in size. Although the electrode model is simple in this new approach, it could be argued that it is sufficient for two rea-
sons. Firstly, the voltage is not recorded on the same electrodes that inject the current. Secondly, the source which is modelled inside (but very close to the boundary) appears to give the similar results to a source modelled on the boundary. It appears therefore for the case of bipolar drive configurations that the application of this simple electrode model is acceptable.

This new approach presents some advantages compared with the first method since

1. At least two forms of the solution can be obtained,

2. By using either the first form or the second form, the convergence can be improved,

3. There is no limitation in terms of increasing the number of terms since the first form does not reach the machine precision as quickly as the second form does.

However, the convergence of both forms is very slow when small electrodes are used (as described earlier in this chapter). This prompted an investigation of the use of accelerating convergence techniques. Two accelerating convergence techniques were investigated, namely Euler’s transformation and Epsilon’s transformation. Both improved the convergence when small electrodes were used. Other accelerating convergence techniques such as Shank’s transformation (Shank, 1955), Poisson’s transformation (Howell, 1995) and Theta’s transformation (Brezinski, 1978) have been investigated. However no improvements were observed using these methods.

Finally, it was also important to assess the validity of the solution to the Forward Problem for a finite right circular cylinder by comparing it with real data and other solvers. The study presented a boundary voltage profile of an interleaved configuration. It showed that the first form was in good agreement with the BEM and with the 1/r model. It also showed that the first form was closer to the BEM than to the 1/r model. The second study was performed by Metherall (1998) where he compared the voltage profiles taken from a tank phantom filled with a saline solution with the 1/r model and the analytical solution for different planes of electrodes. He concluded that the analytical method was more accurate than the 1/r model he employed.
Chapter 5
Image Reconstruction

In this chapter, EIT images are reconstructed using the analytical Forward Problem combined with the sensitivity method explained in chapter 2. Metherall (1996), (1998) already showed using the 1/r model that images of a three dimensional object could be reconstructed. Therefore, the first aim of this chapter is to show how the analytical expressions described in chapter 4 can be used to reconstructing images using the sensitivity matrix approach.

5.1 Construction of the sensitivity matrix

In chapter 2, we saw that Geselowitz’s theorem states the relationship between the magnitude of the boundary voltage change \( g_p \) resulting from a small conductivity \( c_p \) within the object (Geselowitz, 1971). This relationship is expressed as

\[
g_p = - \int_{\Omega} c_p \nabla \Phi_u \cdot \nabla \Psi_u dv
\]  

(5.1)

where

- \( \nabla \Phi_u \) is the field associated with the uniform conductivity for the first pair of electrodes,
- \( \nabla \Psi_u \) is the field associated with the uniform conductivity for the second pair of electrodes,
- \( g_p \) is the difference between the change voltage and the uniform voltage,
- \( c_p \) is the difference between the change conductivity and the uniform conductivity,
- \( \Omega \) is a closed region.

Differentiating (4.22) or (4.24) with respect to \( \rho, \phi, z \) allows the sensitivity relation for any drive configuration to be determined. The relation (5.1) can be summarized in matrix notation as

\[
g_p = Sc_p
\]  

(5.2)

where \( S \) is the Sensitivity matrix.
\[ S_{ij} = -\int_i \nabla \Phi_u \cdot \nabla \Psi_u dv \]  
(5.3)

where \( i \) refers to the \( i^{th} \) pixel and \( j \) refers to the \( j^{th} \) drive/receive pair of electrodes.

However, for this study an alternative method was used to construct the sensitivity matrix for a two dimensional plane but employing the three dimensional analytical expression. The sensitivity matrix from (5.3) is constructed using a shape function to interpolate the voltage obtained on triangular element nodes in order to give the sensitivity for each triangular element (Murai and Kagawa, 1985), where

\[ S_{ij} = -\int_{v_j} \frac{\nabla \Phi}{I_\Phi} \cdot \frac{\nabla \Phi}{I_\Phi} dv_j = -\frac{1}{I_\Phi I_\Psi} \Phi_c \Phi_c^T + \frac{cc^T}{4A_e} \Psi_e \]  
(5.4)

where
\( \nabla \Phi \) is the gradient voltage for the drive electrodes,
\( \nabla \Psi \) is the gradient voltage for the receive electrodes,
\( I_\Phi \) is the current injected on the drive electrodes,
\( I_\Psi \) is the current injected on the receive electrodes,
\( A_e \) is the area of the element,
\( b \) is the vector coordinates of each node of an element along the x axis,
\( c \) is the vector coordinates of each node of an element along the y axis,
\( T \) is the matrix transposed.

This method offers the flexibility of using different shape sensitivity elements by the appropriate use of different interpolation functions.

### 5.2 The inversion of the sensitivity matrix

In order to form the conductivity images \( (c_p = S^{-1} g_p) \), the sensitivity matrix \( S \) has to be inverted. In the Sheffield algorithm, the matrix can be replaced by a normalized matrix which is known as the forward matrix \( F \) (Barber, 1992). The forward matrix \( (F) \) is computed as

\[ F = G^{-1} S \]  
(5.5)
where
\( F \) is the forward matrix,
\( G^{-1} \) is a diagonal matrix which contains the values of \( g \) that would be measured for an object of uniform conductivity at each receive pair,
\( S \) is the sensitivity matrix.

This boundary voltage vector can be derived by implementing the analytical expressions given above, to calculate the voltages at each of the receive electrode sites for a given drive pair. The vector \( g_n \) is then given by the appropriate voltage difference between electrodes which constitute an electrode pair.

The matrix \( F \) must also be inverted in order to compute the normalized conductivity. Unfortunately \( F \), like \( S \), is also ill-conditioned and hence its inversion to form \( F^{-1} \) will require the application of regularization methods. One such method is to use the truncated singular-value decomposition (SVD) technique which is presented in the next section.

### 5.3 Regularization Method: Singular Value Decomposition (SVD) Technique

The singular value decomposition matrix was introduced in 1873 by the Italian mathematician Eugenio Beltrami for solving the problem of the diagonalization of a bilinear form (Beltrami, 1873). A more complete treatment was published a year later by Camille Jordan (Jordan, 1874). An interesting history of SVD can be found in (Stewart, 1993). Since the proof and the development of SVD is beyond the scope of this thesis, this section will just introduce SVD.

Let us consider a system of real linear equations expressed in a matrix form as:

\[
Ax = b
\] (5.6)
where $A$ contains $m$ rows and $n$ columns and $m \geq n$.

If we were to use a least square method, the above system will be to find the vector $x$ which minimises the sum of squares:

$$\|b - Ax\|^2$$  \hspace{1cm} (5.7)

If this system is non-singular, a single vector $x$ will satisfy (5.7) and for $m \geq n$, the inverse $A^{-1}$ of $A$ is given by:

$$A^{-1} = (A^t A)^{-1} A^t$$  \hspace{1cm} (5.8)

where $A^t$ denotes the matrix transpose of $A$.

The least square method is applicable as long as the matrix $(A^t A)$ is not singular. As Golub and Van Loan (Golub and Van Loan, 1989) described in their book, problems appear when $(A^t A)$ becomes nearly singular since an infinite number of $x$ vectors exist which minimises (5.7). The Singular Value Decomposition technique modifies the rank and thus the condition of the matrix in order to yield the minimum least square inversion of $A$. The standard formulation for SVD can be found in many books as:

\[ A = U \Sigma V^* \]  \hspace{1cm} (5.9)

Here $V^*$ denotes the adjoint of the matrix $V$. When the standard algorithms of SVD are applied to the case of a real matrix and in such a case, the matrix $V$ is also real, so that
the adjoint matrix $V^*$ coincides with the transposed matrix $V^t$. Therefore, equation (5.9) can be re-written as:

$$ A = U \Sigma V^t $$

where

- $U$ is the orthogonal matrix,
- $V$ is the orthogonal matrix,
- $\Sigma$ is a rectangular matrix of the same size of $AA^t$.

Furthermore, the columns of $U$ are the Eigenvectors of $AA^t$, the columns of $V$ are the Eigenvectors of $A^tA$, and the non-zero entries of $\Sigma$ are the nonnegative square roots of the Eigenvalues of either $AA^t$ or $A^tA$. These are called the singular values of $A$ and are organised in a descending order

$$ \lambda_{11} \geq \lambda_{22} \geq ... \geq \lambda_{nn} \geq 0 $$

From the above definition, the rank and the condition number of the matrix $A$ can be defined as

1) The rank of the matrix $A$ is the number of non-zero singular values (if $A$ has a full rank, then there are $n$ singular values).

2) The condition number $k$ of a matrix is defined as the ratio of the largest (in magnitude) to the smallest singular value.

For a matrix of less than full rank, the singular values will decrease in magnitude at the singular value $\lambda_i$ where $i > r$. Furthermore, noise is introduced during the calculation of $AA^t$ or $A^tA$, resulting in some very small singular values which are meaningless.

Since the inverse of an orthogonal matrix is its transpose, the inverse of matrix $U$ and $V$ are found by computing their transpose. Furthermore, the inverse of a diagonal matrix
like $\Sigma$ is found by replacing each diagonal entry by its reciprocal. Therefore for a well conditioned non-singular matrix $A$, its inverse is given as:

$$A^{-1} = V\Sigma U^T$$

(5.12)

However, when the problem is ill conditioned like in EIT, we find that

1. The matrix is ill-conditioned and many of the singular values are very small,
2. The matrix is singular and thus singular values with positions that exceed the known rank are meaningless.

### 5.4 Image Reconstruction Algorithm

When the matrix is singular, its inversion can be found by excluding the meaningless singular values from the inversion process. This process is known as regularization in the form of the Moore-Penrose pseudo-inverse or generalised matrix inverse. The inversion of the singular matrix $A$ is expressed as:

$$A^+ = V\Sigma^+ U^T$$

(5.13)

Where

$A^+$ is the minimal least squares inverse of $A$,

$$\Sigma_i^+ = \begin{cases} \frac{1}{\sigma_i} & \text{if } i \leq r \\ 0 & \text{if } i > r \end{cases}$$

Since small singular values are associated with most ill-posed measurements, the inclusion of the small singular values for ill-conditioned matrices in their inversion will lead to problems in obtaining a stable solution. As the measurements are corrupted by noise, the corresponding large reciprocal values in $\Sigma^+$ will tend to amplify the noise, and the solution will be swamped. It is therefore likely that regularization of the small singular values will be necessary for ill-conditioned matrices. By the use of (5.13), the small singular values are excluded from the inverse by treating any singular values less
than a predetermined threshold as zero. Such regularization is known as Truncated SVD (TSVD) and this can be considered as a filtering operation in which the filter coefficients are either 1’s or 0’s.

Consider the product $AA^t$ in terms of SVD of $A$, this can be written as:

$$AA^t = U\Sigma V^+ V\Sigma U^+ = U\Sigma^2 U^+$$  \hspace{1cm} (5.14)

Knowing that the transpose of a product is given by the product of the transposed factors in reverse order, and $V^t V = I$, where $I$ is the identity matrix. The pseudo inverse is:

$$(AA^t)^+ = U\Sigma^{-2} U^t$$  \hspace{1cm} (5.15)

By multiplying (5.15) by the transpose of $A$, we get:

$$A^+ = A^t(AA^t)^+$$  \hspace{1cm} (5.16)

and this result is identical to that given by equation (5.13). The Moore-Penrose pseudo-inverse can therefore also be written as:

$$A^+ = A^t(AA^t)^+$$  \hspace{1cm} (5.17)

and the inversion problem is now reduced to finding the pseudo-inverse of $AA^t$, which is smaller than $A$ and is square ($m$ by $m$).
If we replace $A$ by our sensitivity matrix $F$, it can be seen that the sensitivity reconstruction algorithm is of similar form to the filtered back-projection scheme of Barber (1990):

\[
\begin{align*}
    c_n &= F^t (FF^t)^+ g_n \text{ (sensitivity method)} \quad (5.18) \\
    c_n &= B(BF)^+ g_n \text{ (back-projection method)} \quad (5.19)
\end{align*}
\]

Therefore, the back-projection matrix $B$ can be replaced by the transposed forward matrix $F^t$ (Avis, 1993). Furthermore, this expression is valid for both two dimensions and three dimensions. Conversely, $B$ approach can only exist in two dimensions.

The next two sections of this chapter present some studies using both forms (equation (4.22) and equation (4.24)) of the analytical solution to the Forward Problem for a finite right circular cylinder presented in the previous chapter. The first study shows that images of an object can be reconstructed using the first form for an interleaved configuration. The plot of the singular values of the $FF^t$ matrix is also presented. The second study shows reconstructed images of an object using the second form when an adjacent configuration is used. The plot of the singular values of the $FF^t$ matrix will also be presented. In order to investigate the difference between two dimensional images and three dimensional images, the plot of the singular values for the two dimensional case will be added to the plot of the singular values for the three dimensional case.

## 5.5 Reconstruction of images using the first form
5.5.1 Parameters of the circular cylinder and electrodes

The analytical solution given in (4.22) was applied to a homogeneous finite right circular cylinder with a radius of 1.0 cm and a height of 2.0 cm. A current of 0.002 mA was applied to the cylinder. The electrical conductivity was isotropic and uniform (1.0 $S/cm$). The square electrodes had a height of 0.06 cm and width of 0.06 cm. Images were reconstructed using 32 square electrodes placed on the curved boundary of the cylinder in an interleaved configuration (16 drive/16 receive) (see figure 5.1). The 32 electrodes were placed close to the top of the cylinder at $z=1.75$ cm (see figure 5.1).

5.5.2 Methods

Two small spheres were placed inside the circular cylinder. The first sphere was placed at a position of 1/4 of the circular cylinder’s radius from the centre and at a height $h/2$ above the mid-plane of the cylinder’s half length (see figure 5.1). The second sphere was placed close to the boundary at 0.75 cm from the center and in the same plane as the first sphere. Both spheres had the same size.

Figure 5.1. Position of the two spheres inside the circular cylinder
Two dimensional images were reconstructed using a two dimensional mesh of 192 elements. $F$ matrices for a three dimensional cylinder have been constructed using the appropriate Forward Problem Solver. The singular value spectra of $(F^t F)$ for the three dimensions have been analyzed as well as their singular vectors. The boundary data have been generated by using a Boundary Element Method (BEM) solver in order to avoid inverse crimes (Colton and Kress, 1992), by truncating the inversion of the matrix at a rank of 85. The mesh was placed inside the plane of the 32 electrodes and two images had been successfully reconstructed using equation (5.18).

5.5.3 Results

Figure 5.3 shows the plot of the singular value spectra of $(F^t F)$ for the three dimensional case. Inspection of the singular vectors shows a rapid decrease in the singular values for the three dimensional case.

Figure 5.4 shows the reconstructed image of the first sphere (closer to the centre of the circular cylinder). Figure 5.5 shows the reconstructed image of the second sphere (closer to the boundary of the circular cylinder).
Figure 5.3. Normalised singular values of $FF^t$ matrix for the interleaved configuration

The images have been successfully reconstructed by truncating the $(FF^t)$ matrix at rank of 85 (condition number 316227).

Figure 5.4. Image of an object close to the centre of the cylinder
5.6 Reconstruction of images using the second form

5.6.1 Parameters of the circular cylinder and electrodes

The analytical solution given in (4.24) was applied to a homogeneous finite right circular cylinder with a radius of 0.715 cm and a height of 2.7 cm. A current of 29.841 mA was applied to the cylinder. The electrical conductivity was isotropic and uniform (1.0 "S/cm"). The square electrodes had a height 0.28cm and width of 0.14cm. Images were reconstructed using 16 square electrodes placed on the curved boundary of the cylinder in an adjacent configuration (see figure 5.6). The 16 electrodes were placed at the mid section of the cylinder (see figure 5.7).
5.6.2 Methods

A computer study is presented in this section where a spherical object was placed inside the circular cylinder at a position of $1/4$ th of the circular cylinder’s radius from the centre and at a height $h/2$ above the mid-plane of the cylinder’s half length (see figure 5.7 and figure 5.8). Two dimensional images were reconstructed using the same mesh presented in the previous section. The boundary data have been generated by using the same BEM presented in the previous section. By truncating the inversion of the matrix at a rank of 89 (Condition number : 10000), two types of image have been successfully reconstructed using equation (5.19). The first type of image was reconstructed by placing the mesh inside the plane of electrodes. The second type of image was reconstructed by placing the mesh such that a cross section of the three dimensional object was inside that plane (see figure 5.9)

![Figure 5.6](image_url)

Figure 5.6. Electrodes placed in an adjacent configuration at the mid plane of the circular cylinder
Figure 5.7. Electrodes placed at the mid section of the cylinder. An adjacent drive configuration was used.

Figure 5.8. Cross-section of the cylinder with the position of an object inside (z=0)
5.6.3 Results

Figure 5.10 shows the plot of the singular value spectra of $(FF^T)$ for the three dimensional cases. It also shows the singular value spectra of $(FF^T)$ for a two dimensional disc which has been added to the plot in order to compare the difference between a two dimensional disc and a three dimensional cylinder.
Inspection of the singular vectors shows a rapid decrease in the singular values for both the two dimensional and the three dimensional cases. At low singular value positions the difference between the singular values of the different cases is small. The loci of the plots starts to diverge around the $70^{th}$ singular value position with the three dimensional singular values decaying more slowly than the two dimensional case. It would appear therefore that the three dimensional case is better conditioned than the two dimensional case. Furthermore, this divergence of the singular values loci occurs at condition numbers, which given the signal-to-noise ratio (SNR) performance of existing instrumentation, will allow extra singular vectors to be successfully recruited into the three dimensional image reconstruction process resulting in improved resolution. In addition, analysis of the singular vectors at lower singular-value positions shows that the three dimensional singular vectors recruit more central information than the two-dimensional case.
In other words, the three dimensional image reconstruction algorithm appears to be better conditioned than the two dimensional image reconstruction image algorithm. This concurs with Sylvester and Uhlmann (Sylvester and Uhlmann, 1986) who conducted a mathematical analysis of the inverse problem associated with EIT and concluded that the problem is better posed in domains greater than 2D.

Figure 5.11 shows a reconstructed image using a mesh situated in the plane of electrodes at z=0 (mid-section of the cylinder). Figure 5.12 show a reconstructed image of an object using a mesh situated in a plane (z=h/4) different from the plane of electrodes. The images have been successfully reconstructed by truncating the inversion of the \((FF^T)\) matrix at a rank of 89 (condition number 10000).

Figure 5.11. Image of an object situated at the mid-section of the cylinder
In this chapter, the construction of the sensitivity matrix using the three dimensional analytical Forward Problem expressions was described. The conductivity distribution was found by inverting the resulting sensitivity matrix. In order to invert the sensitivity matrix, regularization methods must be used. In this thesis, the truncated SVD regularization technique was employed. By doing so, two expressions were derived (equation (5.18) and equation (5.19)). While equation (5.19) can only be applied to two dimensions, equation (5.18) can be applied to two dimensions as well as to three dimensions. Therefore, equation (5.18) was used for reconstructing images. In this chapter, some images using simulated data were reconstructed using the second form for an interleaved configuration as well as for an adjacent configuration. The object being imaged was reconstructed at the correct position.

Furthermore, a study of the impact of off-plane on the singular values was performed for the adjacent drive configuration. It reveals that the behaviour of the singular values was
much better behaved in three dimensions than in two dimensions. However, this study should be extended to other data collection methods in order to see if this still holds true.
Chapter 6
Modelling the Forward Problem for an Elliptical Cylinder

One of the most widely studied EIT applications is that of imaging the human thorax. Whilst the human thorax and its contents are a complex three dimensional shape, most EIT image reconstruction algorithms have ignored this and use approximations to the shape of the human thorax ranging from a two dimensional disc, infinite half space (1/r model) to a finite right circular cylinder as reported in this thesis.

However, using such simple geometrical shapes even for the three dimensional models are questionable since the human thorax has a complex shape. For this reason, some research groups have started to investigate the use of numerical solutions such as FEM or BEM to more accurately model the irregular boundary of the thorax. Whilst this is recognised as one way forward, the practical realisation of such an approach presents many challenges. An alternative approach, and this is pursued here, is to investigate the utility of employing simple methods based on analytical methods involving elliptical cylinders whose shapes may be easily changed to determine if such methods improve EIT imaging of the human thorax.

This chapter starts by defining the elliptic coordinates and Poisson’s equation in terms of elliptic coordinates. Then, the derivation of the solution to the Forward Problem for an elliptical cylinder is detailed. The methods employed are similar to those used in previous chapters. The chapter concludes by presenting some results in terms of reconstruction of equipotentials and in terms of comparisons between the analytical solution and the boundary element method.
6.1 Poisson’s equation expressed in elliptic coordinates

The elliptic coordinates are given by

\[
x = l \cosh(u) \cos(v) \quad (6.1)
\]
\[
y = l \sinh(u) \sin(v) \quad (6.2)
\]

where
\[u \geq 0, \ 0 \leq v \leq 2\pi,\]
\[l = \sqrt{a^2 - b^2}\]

where \(a, b\) are defined in figure 6.1.

Furthermore, the eccentricity is defined as

\[e = \frac{l}{a} = \frac{\sqrt{a^2 - b^2}}{a} \quad (6.3)\]

Poisson’s equation expressed in elliptic coordinates becomes:

\[
\nabla^2 \Phi = \frac{1}{l^2(\sinh^2(u) + \sin^2(v))} \left( \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} \right) + \frac{\partial^2 \Phi}{\partial z^2} = -J(r_0) \quad (6.4)
\]

where
\(u, v, z\) are the elliptic coordinates,
\(J(r_0)\) is the current density,
\(r_0\) is the position of the source.
Major axis: \( AA'' = 2a \)
Minor axis: \( BB'' = 2b \)
Foci are positioned at \( L, L' \)
The distance from the centre to one focus point is \( 1 = (a^2 - b^2)^{1/2} \)

Figure 6.1. Elliptic coordinates

\[ c : \text{half length of the cylinder}, \ a : \text{half length of the major axis}, \ b : \text{half length of the minor axis}, \ S_1 : \text{height of the electrode 1}, \ S_2 : \text{height of the electrode 2}, \ W_1 : \text{width of the electrode 1}, \ W_2 : \text{width of the electrode 2} \]

Figure 6.2. Parameters of the elliptical cylinder with two electrodes
6.2 Derivation of the Green’s function for an elliptical cylinder

In order to derive a solution to the Forward Problem, the geometric parameters of our elliptic cylinder must be defined as well as the electrodes. Figure 6.2 describes the geometric parameters and the electrodes.

Equation (6.4) can be solved by the application of Green’s functions. For a solution to equation (6.4), the source needs to be replaced by an impulse function. Therefore, equation (6.4) can be rewritten as

\[ \nabla^2 G(r \mid r_0) = -\delta(r - r_0) \]  

(6.5)

where

- \( G(r \mid r_0) \) is the Green’s function,
- \( r_0 \) is the position of the current source,
- \( r \) is the position at which we seek a solution.

By using the same technique presented in chapter 4, the total solution (\( \Phi \)) at position (\( r \)) is then found by summing the results of each integration over electrodes, as follows:

\[ \Phi(r) = -\sum_{i=1}^{\text{number of electrodes}} \int_{A} \int G(r \mid r_0) J_i(r_0) dr_0 \]  

(6.6)

where \( J_i \) is the current density applied on the \( i^{th} \) electrode,

- \( A \) is the surface area of the \( i^{th} \) electrode.
Equation (6.5) is rewritten in terms of elliptic coordinates as:

\[
\frac{1}{\sqrt{2}l^2(\cosh(2u) - \cos(2v))} \left[ \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2} \right] + \frac{\partial^2 G}{\partial z^2} = -\frac{\delta(u - u_0)\delta(v - v_0)\delta(z - z_0)}{\sqrt{2}l^2(\cosh(2u) - \cos(2v))} \tag{6.7}
\]

where

\[\delta(u - u_0)\delta(v - v_0)\delta(z - z_0)\]

are delta functions, with \(u_0, v_0, z_0\) denoting the current source location.

In order to find the modal coefficients, the Eigenfunctions expansion method is applied. In this technique a solution to equation (6.7) is represented by a series of Eigenfunctions of the associated Eigenvalue problem

\[
\frac{1}{\sqrt{2}l^2(\cosh(2u) - \cos(2v))} \left[ \frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} \right] + \frac{\partial^2 V}{\partial z^2} = 0 \tag{6.8}
\]

with \(V\) satisfying the boundary conditions given by

\[
\left. \frac{\partial V}{\partial z} \right|_{z = \pm c} = 0 : \text{there is no current flow across the ends of the cylinder,} \\
\left. \frac{\partial V}{\partial z} \right|_{z = \pm a} = 0 : \text{there is no current flow across the edges of the cylinder} \tag{except at the electrodes}, \\
V(\phi + 2\pi) = V(\phi) : \text{the potential must be periodic.}
\]

The even and the odd Eigenfunctions of the above equation are:

\[
Ve_{n,m,r} = Je_n(h\epsilon_{nm}, \cosh(u))Se_n(h\epsilon_{nm}, \cos(v)) \cos(\beta_r(z - c)) \\
Vo_{n,m,r} = Jo_n(h\delta_{nm}, \cosh(u))So_n(h\delta_{nm}, \cos(v)) \cos(\beta_r(z - c)) \tag{6.9}
\]

where
\[ \beta_r = \frac{r}{2}, \]
\[ J_{e_{n}} \text{ is the } r^{th} \text{ radial first kind even Mathieu’s function}, \]
\[ J_{o_{n}} \text{ is the } r^{th} \text{ radial first kind odd Mathieu’s function}, \]
\[ S_{e_{n}} \text{ is the } r^{th} \text{ circumferential first kind even Mathieu’s function}, \]
\[ S_{o_{n}} \text{ is the } r^{th} \text{ circumferential first kind odd Mathieu’s function}, \]
\[ h_{e_{nm}} \text{ is the Eigenvalue satisfying the boundary condition,} \]
\[ \frac{\partial V}{\partial \theta} \big|_{u=u_1} = 0, \text{ and called } m \text{th root of } J' e_{n}(h, \cosh(u)) = 0, \]
\[ h_{o_{nm}} \text{ is the Eigenvalue satisfying the boundary condition,} \]
\[ \frac{\partial V}{\partial \theta} \big|_{u=u_1} = 0, \text{ and called } m \text{th root of } J' o_{n}(h, \cosh(u)) = 0, \]

More details on Mathieu’s functions can be found in (Alhargan, 1996). Once the Eigenfunctions are known, the Green’s function can be expressed as the sum of all the Eigenfunctions. Therefore, the Green’s function for the even case is the sum of all the even Eigenfunctions given by \( V_{e_{nm}} \) and is written as

\[ G_e = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \alpha_{n,m,r} J_{e_{n}}(h_{e_{nm}}, \cosh(u)) S_{e_{n}}(h_{e_{nm}}, \cos(v)) \cos(\beta_r(z - c)) \quad (6.10) \]

Substituting equation (6.10) into equation (6.7),

\[ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \left( k_{e_{nm}}^2 + \beta_r^2 \right) \alpha_{n,m,r} J_{e_{n}}(h_{e_{nm}}, \cosh(u)) S_{e_{n}}(h_{e_{nm}}, \cos(v)) \]
\[ \times \cos(\beta_r(z - c)) = \frac{\delta(u - u_0) \delta(v - v_0) \delta(z - z_0)}{\pi^2 (\cosh(2u) - \cos(2v))} \quad (6.11) \]

where

\[ h_{e_{nm}} = l k_{e_{nm}}. \]

By multiplying both sides with

\[ J_{e_{n}}(h_{e_{nm}}, \cosh(u)) S_{e_{n}}(h_{e_{nm}}, \cos(v)) \cos(\beta_r(z - c)) \frac{l^2}{2} (\cosh(2u) - \cos(2v)) \quad (6.12) \]
and integrating within the limit of the elliptical cylinder by using the following relationships

\[ T_{enm} = \int_{0}^{u_1} \int_{0}^{2\pi} J^2_n(h\epsilon_{nm}, \cosh(u)) Se^2_n(h\epsilon_{nm}, \cos(v))(\cosh(2u) - \cos(2v)) \, du \, dv \]

where \( \sigma_r = \begin{cases} 1 & r = 0 \\ 2 & r \neq 0 \end{cases} \)

The following expression is obtained

\[ (k\epsilon^2_{nm} + \beta^2_r) Ae_{n,m,r} T_{enm} t^2 \frac{2c}{2\sigma_r} = J_n(h\epsilon_{nm}, \cosh(u_0)) Se_n(h\epsilon_{nm}, \cos(v_0)) \cos(\beta_r(z_0 - c)) \]

Using the orthogonality between Eigenfunctions, \( Ae_{n,m,r} \) is determined as

\[ Ae_{n,m,r} = \frac{\sigma_r J_n(h\epsilon_{nm}, \cosh(u_0)) Se_n(h\epsilon_{nm}, \cos(v_0)) \cos(\beta_r(z_0 - c))}{c(k\epsilon^2_{nm} + \beta^2_r)t^2 T_{enm}} \]

where

\[ \beta_r = \frac{\pi r}{2c} \]

\( J_n \) is the \( n^{th} \) radial first kind even Mathieu’s function,

\( Se_n \) is the \( n^{th} \) circumferential first kind even Mathieu’s function,

\[ k\epsilon^2_{nm} = \frac{h^2}{2\pi^2} \]

\( h\epsilon_{nm} \) is the Eigenvalue satisfying the boundary condition,

\[ \frac{\partial \psi}{\partial r} \bigg|_{u = u_1} = 0, \text{ and called } m^{th} \text{ root of } J' n(h, \cosh(u)) = 0, \]

\[ T_{enm} = \int_{0}^{u_1} \int_{0}^{2\pi} J^2_n(h\epsilon_{nm}, \cosh(u)) Se^2_n(h\epsilon_{nm}, \cos(v)) \times (\cosh(2u) - \cos(2v)) \, du \, dv. \]

By replacing the coefficients \( Ae_{n,m,r} \) into equation (6.10), the even Green’s function for the even part becomes
where

\[ Ge(u, v, z | u_0, v_0, z_0) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \sigma_r J_n(h \epsilon_{nm}, \cosh(u)) S_n(h \epsilon_{nm}, \cos(v)) \]

\[ \times J_n(h \epsilon_{nm}, \cosh(u_0)) S_n(h \epsilon_{nm}, \cos(v_0)) \]

\[ \times \cos(\beta_r(z_0 - c)) \cos(\beta_r(z - c)) \]

(6.16)

The potential \( (\Phi_1) \) for the even case due to the first electrode is found by replacing \( Ge(u, v, z | u_0, v_0, z_0) \) from equation (6.16) into equation (6.6) and integrating over the surface of the rectangular electrode which has a length \( S_1 \) and a width \( W_1 \) with \( \Delta_1 \) being the subtended angle determined as

\[ \Delta_1 = \tan^{-1}\left( \frac{W_1}{2 \sqrt{\cosh^2(u_1) + \sinh^2(u_1) \sin^2(v_1)}} \right) \]

which becomes

\[ \Phi_1 = \int_{v_1 - \Delta_1}^{v_1 + \Delta_1} \int_{z_1 - \frac{z_0}{2}}^{z_1 + \frac{z_0}{2}} J Ge(u, v, z | u_0, v_0, z_0) I(S_1^2(u_1) + \sin^2(v_0))^\frac{1}{2} d\epsilon_0 dz_0 \]

(6.17)

where

\[ IS_n(h, v_1, \Delta_1, u_1) = \int_{\epsilon_1 - \Delta_1}^{\epsilon_1 + \Delta_1} S_n(h, \cos(v_0)) (\sinh^2(u_1) + \sin^2(v_0))^\frac{1}{2} d\epsilon_0 \]

However, there is a special case which has to be taken into account when \( n \) and \( m \) approach zero. By knowing
Equation (6.18) reduces to

\[ \Phi_e(u, v, z) = -\sum_{r=1}^{\infty} \frac{4J_1}{c\pi l^2 \beta_r^2 \sinh(2u_1)} IS e_0(0, v_1, \Delta_1, u_1) \cos(\beta_r(z_1 - c)) \times \cos(\beta_r(z - c)) \sin(\beta_r \left(\frac{S_1}{2}\right)) \]  

(6.20)

The complete even solution is

\[ \Phi_e(u, v, z) = -\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{2J_1 \sigma_r J_e_n(he_{nm}, \cosh(u))Se_n(he_{nm}, \cos(v))}{c\beta_r(k^2 + \beta_r^2)IT e_{nm}} \times J_e_n(he_{nm}, \cosh(u))IS e_n(he_{nm}, v_1, \Delta_1, u_1) \times \cos(\beta_r(z_1 - c)) \cos(\beta_r(z - c)) \sin(\beta_r \left(\frac{S_1}{2}\right)) \]  

(6.21)

where

\[ \beta_r = \frac{e_0 v}{2z}, \]
\[ J_e_n \text{ is the } n^{th} \text{ radial first kind even Mathieu's function,} \]
\[ Se_n \text{ is the } n^{th} \text{ circumferential first kind even Mathieu's function,} \]
\[ ke_{nm}^2 = \frac{he_{nm}^2}{l^2}, \]
\[ he_{nm} \text{ is the Eigenvalue satisfying the boundary condition,} \]
\[ \frac{\partial V}{\partial \nu} \bigg|_{u=u_1} = 0, \text{ and called mth root of } J_e_n(h, \cosh(u)) = 0, \]
\[ Te_{nm} = \int_0^{2\pi} J_e_{nm}(he_{nm}, \cosh(u))Se_{nm}(he_{nm}, \cos(v)) \times (\cosh(2u) - \cos(2v))du, \]
\[ IS e_n(h, v_1, \Delta_1, u_1) = \int_{v_1 - \Delta_1}^{v_1 + \Delta_1} Se_n(h, \cos(v_0))(\sinh^2(u_1) + \sin^2(v_0))^2dv_0. \]

The odd part is obtained in the same manner and is given by
where

\[ \beta_r = \frac{r \pi}{2a}, \]

\( J_{n} \) is the \( n^{th} \) radial first kind odd Mathieu’s function,

\( S_{nm} \) is the \( n^{th} \) circumferential first kind odd Mathieu’s function,

\[ k_{nm}^2 = \frac{e_{nm}^2}{a^2}, \]

\( e_{nm} \) is the Eigenvalue satisfying the boundary condition,

\[ \frac{\partial J_n}{\partial u} \bigg|_{u=0} = 0, \text{ and called mth root of } J_n(h, \cosh(u)) = 0, \]

\[ T_{nm} = \int_{0}^{2\pi} J_n^2(h_{nm}, \cosh(u)) S_{nm}^2(h_{nm}, \cosh(v)) \]

\[ \times (\cosh(2u) - \cosh(2v)) du dv, \]

\[ IS_{nm}(h, v_1, \Delta_1, u_1) = \int_{v_1-\Delta_1}^{v_1+\Delta_1} S_{nm}(h, \cos(v_1)(\sinh^2(u_1) + \sin^2(v_1)))^{1/2} dv_1. \]

The total potential is found by summing the even and the odd part, i.e:

\[ \Phi_1(u, v, z) = \Phi_e(u, v, z) + \Phi_o(u, v, z) \quad (6.23) \]

### 6.3 Results

Two sets of results will be presented in this chapter. The first set of results shows a comparison between the BEM and the analytical solution for the three different eccentricities at seven different points. The second set of results shows the reconstruction of the equipotentials on the surface of the elliptical cylinder for the three eccentricities. In both studies, the elliptical cylinder had a height of 2.7 cm and a minor axis of 0.715 cm. The major axis was varied to produce eccentricities between 0.15, 0.30 and 0.40. Two electrodes were attached on the elliptical cylinder and placed in a polar configuration on
the minor axis. The electrodes had a width 0.14cm and a height of 0.26cm (see figure 6.3). A current of 48.8525 mA was used. The conductivity was uniform \(1.0 \, S/cm^{-1}\).

![Diagram of an elliptical cylinder with parameters and a current source](image)

Figure 6.3. Parameters of the elliptical cylinder where two electrodes are placed in a polar configuration

### 6.3.1 Comparisons between BEM and the analytical solution

The potentials on seven points using three different eccentricities \(e=0.15\), \(e=0.30\) and \(e=0.40\) were compared between the BEM and the analytical solution. Figure 6.4 shows the position of the seven points on the boundary of the elliptical cylinder.
Figure 6.4. Position of seven points on the boundary of the elliptical cylinder

Figure 6.5 shows the comparison of potentials on the seven points between the BEM and the analytical solution for an eccentricity of $e=0.15$.

Figure 6.6 shows the comparison of potentials on the seven points between the BEM and the analytical solution for an eccentricity of $e=0.30$. 
Figure 6.6. Comparison of potentials between the analytical expression and the BEM for seven points using an eccentricity of 0.30

Figure 6.7 shows the comparison of potentials on the seven points between the BEM and the analytical solution for an eccentricity of e=0.40.

Figure 6.7. Comparison of potentials between the analytical expression and the BEM for seven points using an eccentricity of 0.40

As the eccentricity increases, so do the differences in terms of potentials between the BEM and the analytical solution particularly for points close to the electrodes. Only
small differences appear between BEM and the analytical result for points far away from
the electrodes. The reason is mainly due to the model of the electrodes employed. In
the BEM, the voltage is constant on the total surface of the electrodes, whilst with the
analytical solution, the voltage is not constant since it presents a peak at the center of the
electrodes and is smaller on the edges of the electrodes. As the size of the electrodes are
quite large, this difference between the BEM and the analytical solution on the electrodes
is more important.

6.3.2 Reconstruction of the equipotentials

The equipotentials are reconstructed on the surface of the elliptical cylinder at different
eccentricities

Figure 6.8 shows that the equipotentials have the right shape for an eccentricity of 0.15.

Figure 6.8. Reconstruction of the equipotentials on the major axis at an eccentricity
of 0.15

Figure 6.9 shows also that the equipotentials are well reconstructed using an eccentricity
of 0.30.
Figure 6.9. Reconstruction of the equipotentials on major axis for an eccentricity of 0.30

Figure 6.10 shows the reconstruction of the equipotentials for an eccentricity of 0.40 and from a different point of view.

Figure 6.10. Reconstruction of the equipotentials on the major axis using an eccentricity of 0.40
6.4 Discussion

It can be seen that the equipotentials have the expected shape for the different eccentricities. Furthermore, the analytical solution results are very similar to those obtained by the use of the BEM. However, more work needs to be done in order to establish:

1. if convergence is completely reached,
2. how these boundary profiles match human data,
3. how these boundary profiles affect image reconstructions,
4. how to improve the electrode model.

Nevertheless for the first time in EIT (Kleinermann et al, 1997), a three dimensional analytical solution to the Forward Problem has been found for an elliptical circular cylinder which models the human thorax in a better way.
Chapter 7

Preliminary Results for Multi-frequency EIT

So far the work presented in this thesis has been based on the application of quasi-static conditions to the governing equations. In other words, the Forward Problem in EIT (and in MEIT) assumes that the quasi-static conditions are valid (see chapter 2). This has the advantage of simplifying Maxwell’s equations since they collapse to Poisson’s equation (or Laplace’s equation). However, medical research groups in MEIT have constructed systems capable of operating at or beyond 1 MHz and this raises the question of the validity of the quasi-static conditions employed. It is therefore important to know when these quasi-static conditions break down and to understand the resulting differences between measured frequency dependent data and frequency independent models used in the image reconstruction algorithms.

The first section of this chapter will investigate the validity of the quasi-static conditions for different tissues and will propose new governing equations for MEIT as well as an analytical solution to the Multi-frequency dependent Forward Problem for MEIT.

7.1 Multi-frequency EIT

In chapter 2, the governing equations for EIT based on Maxwell’s equations for time-harmonic were derived. It was shown that Maxwell’s equations given by

\begin{align*}
\nabla \times E &= -i\omega \mu H \\
\nabla \times H &= \gamma E
\end{align*}

(7.1) (7.2)
where

\( E \): electric density field,
\( H \): magnetic density field,
\( \omega \): angular frequency,
\( \mu \): permeability,
\( \gamma = (\sigma + i\omega) \),
\( \sigma \): conductivity,
\( \epsilon \): permittivity.

could be simplified since EIT traditionally operated at low frequency (1 kHz-100kHz). Furthermore, in this range of frequencies, the quasi-static assumptions are defined as

1. \( \mu \approx \mu_0 \) for biological tissue (and especially at low frequency),
2. The effect of magnetic induction that causes the induced electric field can be neglected,
3. The capacitive effect at low frequency can be neglected.

By applying the quasi-static conditions, the full Maxwell’s equations simplify to:

\[
\nabla \times E = 0 \quad (7.3)
\]
\[
\nabla \times H = \sigma E \quad (7.4)
\]

Using the simplified full Maxwell’s equations, the governing equation for EIT was found as

\[
\nabla \cdot (\sigma \nabla \phi) = 0 \quad (7.5)
\]

where

\( \nabla \): Laplacian operator,
\( \phi \): complex valued electric potential,
\( \sigma \): where conductivity.

It was also shown that the magnetic induction can be neglected if
\[
\omega \mu \sigma L_c^2 (1 + \frac{\omega \epsilon}{\sigma}) \ll 1
\]

where \(L_c\) is a characteristic distance over which \(E\) varies significantly,

and the capacitive effect can be neglected if

\[
\frac{\omega \epsilon}{\sigma} \ll 1
\]

Recently, several groups (Brown, 1999), (Osypka and Gersing, 1995), (Rigaud et al., 1995), (Jossinet, 1998) have started to use frequencies up to 10 MHz. This obviously raises the question about the validity of the above quasi-static conditions (7.6), (7.7) for different tissues. As a consequence, there is a need to test these assumptions across a range of frequencies and for different tissues. Furthermore, it is important to know what the difference would be between the models based on the quasi-static conditions (today EIT models) and new models which remove the quasi-static restrictions. The next section presents a first study which looks at the order of magnitude of the quasi-static conditions across a range of frequencies and for different tissues.

### 7.1.1 Magnitude of the quasi-static conditions for different tissues across a range of frequencies

In order to investigate the quasi-static assumptions, the permittivity values and the conductivity values previously published (Metherall, 1998) for different tissues within the frequency range (1 kHz to 10 MHz) will be used for computing these assumptions. These published material property values detail the permittivity and the conductivity (computed) at five different frequencies and for nine samples of living tissue which were either human or bovine. Table 8.1 and table 8.2 show the values of the permittivity and conductivity for these nine samples of living tissue.
The first static condition (7.6) with the characteristic distance of 0.4m (= diameter of the chest) and the second static condition are applied to these results (table 8.1 and table 8.2). Table 8.3, table 8.4, table 8.5, table 8.6 and table 8.7 show the values of these static condition at 1 kHz, 10 kHz, 100 kHz, 1 MHz and 10 MHz for nine samples of living tissue.

Table 7.1. Conductivity ($Sm^{-1}$) at five different frequencies where B=Bovine, H=Human. Adapted from Metherall (1998).

<table>
<thead>
<tr>
<th>Material</th>
<th>Species</th>
<th>1kHz</th>
<th>10 kHz</th>
<th>100 kHz</th>
<th>1 MHz</th>
<th>10 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain</td>
<td>B</td>
<td>0.1</td>
<td>0.13</td>
<td>0.15</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>Liver</td>
<td>B</td>
<td>0.04</td>
<td>0.05</td>
<td>0.09</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>Kidney</td>
<td>B</td>
<td>0.12</td>
<td>0.15</td>
<td>0.2</td>
<td>0.3</td>
<td>0.5</td>
</tr>
<tr>
<td>Muscle (across)</td>
<td>B</td>
<td>0.3</td>
<td>0.35</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>Muscle (along)</td>
<td>B</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>Lung (inflated)</td>
<td>B</td>
<td>0.05</td>
<td>0.06</td>
<td>0.08</td>
<td>0.1</td>
<td>0.20</td>
</tr>
<tr>
<td>Uterus</td>
<td>H</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>Skin</td>
<td>H</td>
<td>7x10^{-4}</td>
<td>4x10^{-3}</td>
<td>6x10^{-2}</td>
<td>3x10^{-1}</td>
<td>4x10^{-4}</td>
</tr>
<tr>
<td>Adipose Tissue</td>
<td>H</td>
<td>0.022</td>
<td>0.023</td>
<td>0.023</td>
<td>0.24</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 7.2. Relative permittivity at five different frequencies where B=Bovine, H=Human. Adapted from Metherall (1998).

<table>
<thead>
<tr>
<th>Material</th>
<th>Species</th>
<th>1kHz</th>
<th>10 kHz</th>
<th>100 kHz</th>
<th>1 MHz</th>
<th>10 MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain</td>
<td>B</td>
<td>2x10^5</td>
<td>2x10^4</td>
<td>4x10^3</td>
<td>1x10^3</td>
<td>3x10^2</td>
</tr>
<tr>
<td>Liver</td>
<td>B</td>
<td>9x10^4</td>
<td>3x10^4</td>
<td>1x10^4</td>
<td>2x10^3</td>
<td>2x10^2</td>
</tr>
<tr>
<td>Kidney</td>
<td>B</td>
<td>2x10^5</td>
<td>4x10^4</td>
<td>1x10^4</td>
<td>2x10^3</td>
<td>4x10^2</td>
</tr>
<tr>
<td>Muscle (across)</td>
<td>B</td>
<td>6x10^5</td>
<td>3x10^4</td>
<td>1x10^4</td>
<td>2x10^3</td>
<td>1x10^2</td>
</tr>
<tr>
<td>Muscle (along)</td>
<td>B</td>
<td>1x10^6</td>
<td>3x10^4</td>
<td>2x10^3</td>
<td>4x10^2</td>
<td>2x10^2</td>
</tr>
<tr>
<td>Lung (inflated)</td>
<td>B</td>
<td>1x10^5</td>
<td>2x10^4</td>
<td>3x10^3</td>
<td>6x10^2</td>
<td>2x10^2</td>
</tr>
<tr>
<td>Uterus</td>
<td>H</td>
<td>1x10^6</td>
<td>2x10^4</td>
<td>3x10^3</td>
<td>1x10^3</td>
<td>3x10^2</td>
</tr>
<tr>
<td>Skin</td>
<td>H</td>
<td>4x10^4</td>
<td>3x10^4</td>
<td>2x10^3</td>
<td>2x10^3</td>
<td>2x10^2</td>
</tr>
<tr>
<td>Adipose Tissue</td>
<td>H</td>
<td>1x10^4</td>
<td>4x10^3</td>
<td>5x10^3</td>
<td>2x10^3</td>
<td>1x10^3</td>
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</tbody>
</table>
Table 7.3. Magnitude of the capacitive effects and the magnetic induction at 1 kHz.

<table>
<thead>
<tr>
<th>Material</th>
<th>Species</th>
<th>Capacitive Effects</th>
<th>Magnetic induction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain</td>
<td>B</td>
<td>0.111265004</td>
<td>0.000140367</td>
</tr>
<tr>
<td>Liver</td>
<td>B</td>
<td>0.125173129</td>
<td>5.69577E-05</td>
</tr>
<tr>
<td>Kidney</td>
<td>B</td>
<td>0.092720855</td>
<td>0.000168563</td>
</tr>
<tr>
<td>Muscle (across)</td>
<td>B</td>
<td>0.111265004</td>
<td>0.000421161</td>
</tr>
<tr>
<td>Muscle (along)</td>
<td>B</td>
<td>0.111265004</td>
<td>0.000701936</td>
</tr>
<tr>
<td>Lung (inflated)</td>
<td>B</td>
<td>0.111265004</td>
<td>7.01936E-05</td>
</tr>
<tr>
<td>Uterus</td>
<td>H</td>
<td>0.139081255</td>
<td>0.000576605</td>
</tr>
<tr>
<td>Skin</td>
<td>H</td>
<td>3.179000106</td>
<td>3.69566E-06</td>
</tr>
<tr>
<td>Adipose Tissue</td>
<td>H</td>
<td>0.025287501</td>
<td>2.84956E-05</td>
</tr>
<tr>
<td>Mean Value</td>
<td></td>
<td>0.116159688</td>
<td>0.00024013</td>
</tr>
</tbody>
</table>

Table 7.4. Magnitude of the capacitive effects and the magnetic induction at 10 kHz

<table>
<thead>
<tr>
<th>Material</th>
<th>Species</th>
<th>Capacitive Effects</th>
<th>Magnetic induction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain</td>
<td>B</td>
<td>0.085868646</td>
<td>0.001783264</td>
</tr>
<tr>
<td>Liver</td>
<td>B</td>
<td>0.333795011</td>
<td>0.000842498</td>
</tr>
<tr>
<td>Kidney</td>
<td>B</td>
<td>0.145353338</td>
<td>0.002176088</td>
</tr>
<tr>
<td>Muscle (across)</td>
<td>B</td>
<td>0.04768502</td>
<td>0.004632426</td>
</tr>
<tr>
<td>Muscle (along)</td>
<td>B</td>
<td>0.033379501</td>
<td>0.00652739</td>
</tr>
<tr>
<td>Lung (inflated)</td>
<td>B</td>
<td>0.186441673</td>
<td>0.000986548</td>
</tr>
<tr>
<td>Uterus</td>
<td>H</td>
<td>0.027816251</td>
<td>0.003663799</td>
</tr>
<tr>
<td>Skin</td>
<td>H</td>
<td>4.172437639</td>
<td>0.00261376</td>
</tr>
<tr>
<td>Adipose Tissue</td>
<td>H</td>
<td>0.009675218</td>
<td>0.000293372</td>
</tr>
<tr>
<td>Mean Value</td>
<td></td>
<td>0.07365371</td>
<td>0.00251204</td>
</tr>
</tbody>
</table>

Table 7.5. Magnitude of the capacitive effects and the magnetic induction at 100 kHz

<table>
<thead>
<tr>
<th>Material</th>
<th>Species</th>
<th>Capacitive Effects</th>
<th>Magnetic induction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain</td>
<td>B</td>
<td>0.148353338</td>
<td>0.021760682</td>
</tr>
<tr>
<td>Liver</td>
<td>B</td>
<td>0.618139009</td>
<td>0.01839789</td>
</tr>
<tr>
<td>Kidney</td>
<td>B</td>
<td>0.278162509</td>
<td>0.032294293</td>
</tr>
<tr>
<td>Muscle (across)</td>
<td>B</td>
<td>0.139061255</td>
<td>0.05756048</td>
</tr>
<tr>
<td>Muscle (along)</td>
<td>B</td>
<td>0.022253001</td>
<td>0.064571088</td>
</tr>
<tr>
<td>Lung (inflated)</td>
<td>B</td>
<td>0.208621882</td>
<td>0.012214907</td>
</tr>
<tr>
<td>Uterus</td>
<td>H</td>
<td>0.041724376</td>
<td>0.052640805</td>
</tr>
<tr>
<td>Skin</td>
<td>H</td>
<td>1.864416728</td>
<td>0.02163068</td>
</tr>
<tr>
<td>Adipose Tissue</td>
<td>H</td>
<td>0.012094022</td>
<td>0.002940752</td>
</tr>
<tr>
<td>Mean Value</td>
<td></td>
<td>0.181397622</td>
<td>0.031567463</td>
</tr>
</tbody>
</table>
Table 7.6. Magnitude of the capacitive effects and the magnetic induction at 1 MHz

<table>
<thead>
<tr>
<th>Material</th>
<th>Species</th>
<th>Capacitive Effects</th>
<th>Magnetic induction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain</td>
<td>B</td>
<td>0.278162609</td>
<td>0.322942928</td>
</tr>
<tr>
<td>Liver</td>
<td>B</td>
<td>0.566325018</td>
<td>0.393223987</td>
</tr>
<tr>
<td>Kidney</td>
<td>B</td>
<td>0.370883346</td>
<td>0.519554921</td>
</tr>
<tr>
<td>Muscle (across)</td>
<td>B</td>
<td>0.222530007</td>
<td>0.772216789</td>
</tr>
<tr>
<td>Muscle (along)</td>
<td>B</td>
<td>0.037086335</td>
<td>0.786986029</td>
</tr>
<tr>
<td>Lung (inflated)</td>
<td>B</td>
<td>0.333793011</td>
<td>0.18449957</td>
</tr>
<tr>
<td>Uterus</td>
<td>H</td>
<td>0.1112600404</td>
<td>0.70199573</td>
</tr>
<tr>
<td>Skin</td>
<td>H</td>
<td>0.370883346</td>
<td>0.519554921</td>
</tr>
<tr>
<td>Adipose Tissue</td>
<td>H</td>
<td>0.004636042</td>
<td>0.304599663</td>
</tr>
<tr>
<td>Mean Value</td>
<td></td>
<td>0.2085227269</td>
<td>0.498796304</td>
</tr>
</tbody>
</table>

Table 7.7. Magnitude of the capacitive effects and the magnetic induction at 10 MHz

<table>
<thead>
<tr>
<th>Material</th>
<th>Species</th>
<th>Capacitive Effects</th>
<th>Magnetic induction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain</td>
<td>B</td>
<td>0.566325018</td>
<td>5.880359604</td>
</tr>
<tr>
<td>Liver</td>
<td>B</td>
<td>0.370883346</td>
<td>5.195549211</td>
</tr>
<tr>
<td>Kidney</td>
<td>B</td>
<td>0.445060005</td>
<td>9.12779909</td>
</tr>
<tr>
<td>Muscle (across)</td>
<td>B</td>
<td>0.092720836</td>
<td>8.262666643</td>
</tr>
<tr>
<td>Muscle (along)</td>
<td>B</td>
<td>0.158550005</td>
<td>10.24879658</td>
</tr>
<tr>
<td>Lung (inflated)</td>
<td>B</td>
<td>0.566325018</td>
<td>3.932239669</td>
</tr>
<tr>
<td>Uterus</td>
<td>H</td>
<td>0.278162509</td>
<td>9.888267829</td>
</tr>
<tr>
<td>Skin</td>
<td>H</td>
<td>0.278162509</td>
<td>6.488585653</td>
</tr>
<tr>
<td>Adipose Tissue</td>
<td>H</td>
<td>0.022253001</td>
<td>3.228664414</td>
</tr>
<tr>
<td>Mean Value</td>
<td></td>
<td>0.275995009</td>
<td>6.895678687</td>
</tr>
</tbody>
</table>

It can be seen from table 7.3, table 7.4, table 7.5, table 7.6 and table 7.7 that the conductive effects start to rise around 1 MHz. This implies that the displacement current is no longer negligible. A summary of the behaviour of the magnetic induction is shown in figure 7.1 where the mean value of the magnetic induction is plotted against the frequencies.
Figure 7.1. Behaviour of the first static condition 7.6 across the frequencies

Figure 7.1 shows that the induced magnetic field starts to rise rapidly around 1 MHz. Although at 1 MHz, the magnitude is still less than one, it is important to realize that the change of the induced magnetic field from 100 kHz to 1 MHz is significant. Clearly, given these results, the validity of the quasi-static conditions warrants further investigation.

In the above study, only the quasi-static conditions were compared for different tissues across different frequencies. This study does not take into account any kind of specific topology or boundary conditions applied to the imaged object as would be the case in industrial or medical applications of MEIT. It is therefore necessary to be able to perform a study which will include the topology and the appropriate boundary conditions. One approach would be to compute the results from the Forward Problem based on quasi-static conditions for a specific geometrical figure (such as the finite right circular cylinder previously reported) and to compare them with the results computed from a Forward Problem based on non quasi-static conditions for the same geometrical figure with the same boundary conditions.
This requires the Forward Problem using non quasi-static conditions to be derived. In this section, we will first restate Maxwell’s equations without assuming quasi-static conditions and then the full analytical solution to the Forward problem for a finite right circular cylinder will be developed.

This new analytical solution will provide a way for comparing the differences between this new analytical solution based on non quasi-static conditions and for a finite right circular cylinder with the previous analytical solution presented in chapter 4. Indeed, if the results from this new analytical solution are significantly different from the ones generated by the analytical solution (presented in chapter 4), then the new governing equation based on quasi-static conditions would need to be used in the MEIT image reconstruction algorithm.

7.1.2 Governing Equation without quasi-static conditions

One method for checking the validity of the quasi-static conditions at those frequencies would be to find a governing equation which is derived directly from Maxwell’s equations. In chapter 2, the Maxwell’s equations were presented as:

\[ \nabla \times E = -i\omega B \] (7.8)
\[ \nabla \times H = i\omega D + J \] (7.9)
\[ \nabla \cdot D = \rho \] (7.10)
\[ \nabla \cdot J = -i\omega \rho \] (7.11)
\[ \nabla \cdot B = 0 \] (7.12)
\[ B = \mu H \] (7.13)
\[ D = \varepsilon E \] (7.14)

The current \( J \) is composed of the ohmic current (\( J_0 \)) and the current sources (\( J_s \)). In other words,
\[ J = J_0 + J_i \]  
\hspace{1cm} (7.15)

where
\[ J_0 = \sigma E, \]
\[ J_i : \text{current source}. \]

By using the above definition of the current and equation (7.14), equation (7.9) can be re-written as

\[ \nabla \times H = (\sigma + i\omega \epsilon)E + J_i \]  
\hspace{1cm} (7.16)

Since \( \nabla \cdot (\nabla \times A) = 0 \) and \( \nabla \cdot B = 0 \), the magnetic flux \( B \) is defined as

\[ B = \nabla \times A \]  
\hspace{1cm} (7.17)

where \( A \) is the magnetic potential.

By replacing equation (7.17) into equation (7.8), equation (7.8) is re-written as

\[ \nabla \times E = -i\omega \nabla \times A \Rightarrow \nabla \times (E + i\omega A) = 0 \]  
\hspace{1cm} (7.18)

By using the relationship \( \nabla \times \nabla \Phi = 0 \), the electric field can be found to be:

\[ E = -i\omega A - \nabla \Phi \]  
\hspace{1cm} (7.19)

where \( \Phi \) is the scalar potential.

\[ ^9 \text{In chapter 2, the equivalent equation was } E = -\nabla \Phi \text{ since the induced magnetic field was neglected} \]
\hspace{1cm} (1st quasi-static condition)
By maintaining the induced magnetic field and taking the divergence of equation (7.19), equation (7.19) can be re-written as:

$$\nabla \cdot E = \nabla \cdot ( -i\omega A - \nabla \Phi ) = -i\omega (\nabla \cdot A) - \nabla \cdot (\nabla \Phi)$$ (7.20)

By using the Lorentz’s condition which states:

$$\nabla \cdot A = -\sigma \mu \Phi - i\omega \epsilon \mu \Phi$$ (7.21)

Equation (7.20) becomes

$$\nabla \cdot E = i\omega \sigma \mu \Phi - \omega^2 \epsilon \mu \Phi - \nabla^2 \Phi$$ (7.22)

Replacing equation (7.14) into equation (7.10), equation (7.10) becomes:

$$\nabla \cdot E = \frac{\rho}{\epsilon}$$ (7.23)

Replacing equation (7.23) into equation (7.22), equation (7.22) becomes:

$$\nabla^2 \Phi + k^2 \Phi = -\frac{\rho}{\epsilon}$$ (7.24)

where \(k^2 = -i\omega \mu (\sigma + i\omega \epsilon)\) and is called the wave number.

Equation (7.24) is the Helmholtz’s equation for the electric potential and represents the new governing equation for MEIT.
As described at the beginning of this chapter, it is important to investigate the difference at certain frequencies and for a certain living tissue between results obtained using Helmholtz’s equation given by equation (7.24) compared to the results obtained using Poisson’s equation given by equation (2.20). If this difference is large, it implies that the induced magnetic field should not be ignored and therefore, equation (7.24) should be used for that frequency and for that living tissue. In order to be able to study that difference, Helmholtz’s equation must be solved for a given topology. In this thesis a circular cylinder is chosen since it is the same geometrical figure used for deriving a solution to Poisson’s equation (see chapter 4). The next section describes how Helmholtz’s equation can be solved using the same tools used in chapter 4 for solving Poisson’s equation.

### 7.1.3 Solution to the new frequency dependent analytical Forward Problem for a finite right circular cylinder

Helmholtz’s equation is given as

\[
\nabla^2 \Phi + k^2 \Phi = -J(r_0)
\]

(7.25)

where \(J\) is a current source at \(r_0\) in 3D space.

In cylindrical coordinates equation (7.25) can be written as

\[
\nabla_i^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} + k^2 \Phi = -J(r_0)
\]

(7.26)

where \(\nabla_i^2\) is the Laplacian in 2D.
Equation (7.26) can be solved through the application of the Green’s function technique as described in chapter 4. For a solution to equation (7.26), the source needs to be replaced by an impulse function. Therefore, equation (7.26) can be rewritten as

\[
\nabla_i^2 G(r|\mathbf{r}_0) + \frac{\partial^2 G(r|\mathbf{r}_0)}{\partial z^2} + k^2 G(r|\mathbf{r}_0) = -\frac{1}{\rho} \delta(r - r_0) \quad (7.27)
\]

where

- \(G(r|\mathbf{r}_0)\) is the Green’s function,
- \(r_0\) is the position of the current source,
- \(r\) is the position at which we seek a solution.

Once the Green’s functions is found like in chapter 4 and chapter 6, the total solution (\(\Phi\)) at position (\(r\)) is then found by summing the result of each integration over electrodes, as follows:

\[
\Phi(r) = - \sum_i \int_A G(r|\mathbf{r}_0) J_i(\mathbf{r}_0) d\mathbf{r}_0 \quad (7.28)
\]

where

- \(J_i\) is the current density applied on the \(i^{th}\) electrode,
- \(A\) is the surface area of the \(i^{th}\) electrode,
- \(G(r|\mathbf{r}_0)\) is the Green’s function.

**Green’s function of the first form**

The case where two rectangular electrodes are arbitrarily placed on the curved surface of a finite right circular cylinder is considered. The geometry of the finite right circular cylinder with two electrodes attached is exactly the one used to derived the analytical solution to Poisson’s equation for a finite right circular cylinder (see chapter 4, figure 4.1).

Equation (7.27) expressed in cylindrical coordinates is
\[ \frac{\partial^2 G}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial G}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 G}{\partial \phi^2} + \frac{\partial^2 G}{\partial z^2} + k^2 G = -\frac{1}{\rho} \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(z - z_0) \quad (7.29) \]

where
\[ \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(z - z_0) \]
are delta functions,
with \( \rho_0, \phi_0, z_0 \) denoting the current source location.

In order to find the modal coefficients, the Eigenfunctions expansion method is applied.

In this technique a solution to equation (7.29) is represented by a series of Eigenfunctions of the associated Eigenvalue problem:

\[ \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} + k^2 V = 0 \quad (7.30) \]

With \( V \) satisfying the boundary conditions given by
\[ \frac{\partial V}{\partial \rho} \bigg|_{\rho=\pm a} = 0 : \text{there is no current flow across the ends of the cylinder,} \]
\[ \frac{\partial V}{\partial \rho} \bigg|_{\rho=a} = 0 : \text{there is no current flow across the edges of the cylinder (except at the electrodes),} \]
\[ V(\phi + 2\pi) = V(\phi) : \text{the potential must be periodic.} \]

The even solution and the odd Eigenfunctions of the above equation are

\[ V_{e_{n,rm}} = J_n(k_{nm}\rho) \cos n\phi \cos \beta_r(z - c) \quad (7.31) \]
\[ V_{o_{n,rm}} = J_n(k_{nm}\rho) \sin n\phi \cos \beta_r(z - c) \quad (7.32) \]

where
\[ \beta_r = \frac{n\pi}{2a}, \]
\( J_n \) is the \( n^{th} \) Bessel function of the first kind,
\( k_{nm} \) is the Eigenvalues satisfying the boundary conditions,
\[ \frac{\partial V}{\partial \rho} \bigg|_{\rho=a} = 0, \]
and called the \( m^{th} \) root of \( J_n'(ka) = 0 \).
Once the Eigenfunctions are known, the Green’s function can be expressed as the sum of all the Eigenfunctions. Therefore, the Green’s function for the even case is the sum of all the even Eigenfunctions given by $V e_{nm}$ and is written as

$$G_e = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} A_{n,m,r} J_n (k_{nm} \rho) \cos(n\phi) \cos(\beta_r (z - c))$$  \hspace{1cm} (7.33)$$

Substituting (7.33) into (7.29), gives

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} - (k_{nm}^2 + \beta^2_r - k^2) A_{n,m,r} J_n (k_{nm} \rho) \cos(n\phi) \cos(\beta_r (z - c))$$

$$= - \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(z - z_0)$$  \hspace{1cm} (7.34)$$

The coefficients $A_{n,m,r}$ can be evaluated using the orthogonality of the Eigenfunctions. By multiplying both sides of (7.34) with $\rho J_n (k_{nm} \rho) \cos n\phi \cos \beta_r (z - c)$, and integrating within the limits of the cylinder, the coefficients are:

$$A_{n,m,r} = \frac{\sigma_n \sigma_r k_{nm}^2 J_n (k_{nm} \rho_0) \cos(n\phi_0) \cos(\beta_r (z_0 - c))}{2c\pi (k_{nm}^2 + \beta^2_r - k^2) (k_{nm}^2 a^2 - n^2) J_n^2 (k_{nm} a)}$$  \hspace{1cm} (7.35)$$

The odd part is obtained in a similar manner. By combining the even and odd parts, the complete Green’s function becomes

$$G(\rho, \phi, z | \rho_0, \phi_0, z_0) = \frac{\sigma_0 \sigma_0 k_{00}^2 J_0 (k_{00} \rho) J_0 (k_{00} \rho_0)}{2c\pi (k_{00}^2 + \beta^2_0 - k^2) (k_{00}^2 a^2) J_0^2 (k_{00} a)}$$
\[
\Phi_1 = - \int_{\phi_1-\Delta_1}^{\phi_1+\Delta_1} \int_{z_1-S_1/2}^{z_1+S_1/2} J_t G(\rho, \phi, z|\rho_0, \phi_0, z_0) \, ad\phi_0dz_0 \quad (7.38)
\]

Equation (7.37) becomes

\[
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\]

simplifying the terms containing \(k_{00}\), equation (7.36) becomes

\[
G(\rho, \phi, z|\rho_0, \phi_0, z_0) = - \frac{1}{2\pi a^2 k^2} + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r k_{nm}^2 J_n(k_{nm}\rho) J_n(k_{nm}\rho_0) \cos(n(\phi - \phi_0))}{2\pi (k_{nm}^2 + \beta_r^2 - k^2)(k_{nm}^2 a^2 - n^2)} \times \frac{\cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c))}{J_n^2(k_{nm}a)}
\]

+ \sum_{r=1}^{\infty} \frac{\sigma_0 \sigma_r k_{00}^2 J_0(k_{00}\rho) J_0(k_{00}\rho_0)}{2\pi (k_{00}^2 + \beta_r^2 - k^2)(k_{00}^2 a^2 - n^2)} \times \frac{\cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c))}{J_0^2(k_{00}a)}
\]

\[
(7.36)
\]

The Potential using the first form

The potential (\(\Phi_1\)) due to the first electrode is found by replacing \(G(\rho, \phi, z|\rho_0, \phi_0, z_0)\) from equation (7.37) into equation (7.28) and integrating over the surface of the rectangular electrode (see appendix E) which has a length \(S_1\) and a width \(W_1\) with \(\Delta_1\) being the subtended angle determined as \(\Delta_1 = \tan^{-1}(\frac{W_1}{2a})\) is given by
\[
\Phi_1 (\rho, \phi, z) = \frac{J_1 \Delta_1 S_1}{c \pi a k^2} - \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{2a J_1 \sigma_n \sigma_r \Delta_1 k_{nm}^2 J_n (k_{nm} \rho) \cos (n (\phi - \phi_1))}{c \beta_r \pi (k_{nm}^2 + \beta_r^2 - k^2) (k_{nm}^2 a^2 - n^2)} \\
\times \frac{\cos (\beta_r (z - c))}{J_n (k_{nm} a)} \sin c(n \Delta_1) \cos (\beta_r (z_1 - c)) \sin (\beta_r S_1 / 2)
\]

\[
- \sum_{r=1}^{\infty} \frac{4 \Delta_1 J_1}{c \pi a \beta_r} \frac{1}{(\beta_r^2 - k^2)} \cos (\beta_r (z_1 - c)) \cos (\beta_r (z - c)) \times \sin (\beta_r S_1 / 2)
\]

(7.39)

The same method is then applied to the electrode(s), with the desired total potentials being the sum of all the potentials due to the electrodes.

### 7.2 Experimental validation of expression

In order to know if the new analytical solution is consistent with reality, it was important to perform an experiment whereby the analytical solution could be compared with real experimentally obtained measurements. Since the analytical solution requires a cylindrical shape and ideally the material tested to be frequency dependent, it was decided to use a syringe filled of human blood (packed red cells) for validating the new analytical solution. A syringe of 60 ml was modified to accommodate two square electrodes of side length 0.005 m for injecting the current and point electrodes of approximately 0.001 m of diameter in the same plane of the injected electrodes for recording the potentials. All electrodes were made from silver. Figure 7.2 shows a photograph of the syringe and the injected electrodes.
Figure 7.2. Picture of the whole syringe

Figure 7.3. Area of the syringe used where two square electrodes are attached and three point electrodes are placed.
7.2.1 Description of the two experiments

This section describes two experiments. The first experiment was conducted at the University College London in the Department of Physiology. In this experiment, an Hewlett-Packard 4284A impedance analyser is used with two electrodes. Since only two electrodes were used for injecting the current and recording the voltages, it was necessary to extend that experiment to four electrodes where two electrodes are used for injecting the current and the two other electrodes are used for recording the potentials. Since the only system using four electrodes at our disposition was the Sheffield’s pulse system (Waterworth et al., 2000), the second experiment consisted of using such system by repeating the first experiment with four electrodes. The second experiment was conducted at Sheffield Hallamshire Hospital in the Department of Medical Physics and Clinical Engineering.

First experiment involving two electrodes

In order to calibrate the instruments, three test tubes of different lengths (0.07 m, 0.10 m and 0.13 m) were used. The three test tubes had the same diameter of 0.01 m. At each end of each test tubes, a chloride silver disc electrode was placed (see figure 7.4). The liquid was injected through a small hole in the upper end cap until bubbles of air were eliminated.

The three test tubes were firstly filled with a saline solution and measurements were recorded in two-terminal mode using a Hewlett-Packard 4284A impedance analyser from 20 Hz to 1 MHz employing a constant current of 1.0 mA. The calibration was performed by taking the intercept of resistance against tube length, when impedance measurements were made with tube lengths of 0.07 m, 0.10 m and 0.13 m. Figure 7.5 shows the resistance (Ω) according to the test tubes.
Figure 7.4. Test tube with two chlorided silver disc electrodes at each end.

Figure 7.5. Test tubes filled with a saline solution

It can be seen that the resistance at 20 Hz and at 100 Hz for the three test tubes are different from the other frequencies. This is mainly due to the electrode contact impedance.
effect which is high at these two frequencies. Furthermore, it was also possible to calculate the conductivity of the saline solution by knowing the dimensions of each test tube. It was found to have a conductivity of 1.7756 S/m which is roughly the same of the saline solution used which was 0.56% of sodium chloride in 100 ml of distilled water.

By using a transfusion bag taken out of refrigerated storage and allowed to reach room temperature (27 degrees Celsius) over two hours, the three test tubes were then filled of packed red blood cells and measurements were collected with the Hewlett-Packard 4284A impedance analyser from 20 Hz to 1 MHz at a constant current of 0.1 mA. Figure 7.6 shows the resistive part of the impedance (Ω) according to the test tubes.

![Figure 7.6. Test tubes filled with packed red blood cells](image)

Again, the resistive part of the impedance at frequency 20 Hz and 100 Hz are different from the others. This may be due to the electrode contact impedance effect which is high at these two low frequencies.
Therefore, it was decided that only the frequencies around 100 kHz will be used for the syringe since the experiment done on the three test tubes shows that the electrode contact impedance effect is low around 100 kHz.

The syringe had a radius of 0.015 m and was filled of red packed blood cells until the height measured from the top was 0.043 m. Using the two square electrodes (having a size of 0.005 m) attached on its surface in a polar configuration, the syringe was then connected to the two terminal mode using a Hewlett-Packard 4284A for recording the impedance and the phase. A constant current of 0.1 mA was injected through the driving electrodes.

Whilst these results were encouraging, only a two electrode measurements scheme was employed. It was therefore decided to investigate the possibility of using four electrodes measurement method.

**Second experiment involving four electrodes**

The only other multi-frequency EIT hardware at our disposal for performing frequency dependent *in-vitro* studies but employing four electrodes, was the impulse driven transfer impedance system developed at the Department of Medical Physics of the Hallamshire hospital (Waterworth *et al.*, 2000). The pulse system extracts the impedance across the frequency using the following expression:

\[
Z(f) = \frac{FT(Pulse_{tissue})}{FT(Pulse_{cal})} \times \text{value of calibration of electrical circuit} \quad (7.40)
\]

where

- \(Z\) : impedance,
- \(FT\) : Fourier transform,
- \(f\) : frequency.

In our experiment, the calibration pulse data were obtained from the syringe filled with saline solution and the tissue is the blood (packed red blood cells). In order to compare the analytical expression with the pulse system, the following expression based on (7.40) was used:
Preliminary Results for Multi-frequency EIT

\[
\frac{FT(Pulse_{tissue})}{FT(Pulse_{cal})} \times \text{measured value of calibration} = \frac{V(f)_{\text{computed}(tissue)}}{V(f)_{\text{computed}(cal)}} \times \text{computed value of calibration} \tag{7.41}
\]

where
\[V(f)\]: voltage,
\[FT\]: Fourier transform,
\[f\]: frequency.

It was important to normalize the data since the measured value of calibration for the saline solution was unknown and would be most probably different from the calibration computed value. Therefore, equation (7.41) becomes:

\[
\frac{FT(Pulse_{tissue})}{FT(Pulse_{cal})} \times \text{measured value of calibration}_1 = \frac{FT(Pulse_{tissue})}{FT(Pulse_{cal})} \times \text{measured value of calibration}_1 \times \frac{V(f)_{\text{computed}(tissue)}}{V(f)_{\text{computed}(cal)}} \times \text{computed value of calibration}_1 \tag{7.42}
\]

where
\[V(f)\]: voltage,
\[FT\]: Fourier transform,
\[f\]: frequency,
\[1\]: lowest frequency.

Furthermore, the normalization of the data will reduce the error due to inaccuracies of the real electrodes’ position compared to those of the theoretical model. The experiment started by filling the syringe with blood up to a height of 0.047 m. The two injected electrodes (having a size of 0.005 m) were placed in a polar configuration and were driving a current of 0.1 mA. Four point electrodes were placed on the syringe. Measurements were made in the time domain and then transform in the frequency domain by applying a Fourier transform which effectively allowed a measured frequency range from 2 kHz up to 1 MHz. Figure 7.7 shows the syringe with the electrodes. All the measurements were recorded in a room temperature of 22 degrees Celsius.
Figure 7.7. Parameters of the section of the syringe filled of packed red blood cells

Figure 7.8 shows the cross section containing the driving electrodes with the adjacent receivers.

Figure 7.8. Cross section containing the driving electrodes with the adjacent receivers

7.2.2 Debye’s model

In order to use the new analytical solution, a Debye’s model of human blood (Lepelaars, 1997) was used to obtain the different relative permittivity values and conductivity values from 100 kHz to 1 MHz (see figure 7.9).
Figure 7.9. Debye’s model used in the analytical expression where $\sigma$ ($Sm^{-1}$): conductivity and $\epsilon_r$: relative permittivity.

7.2.3 Results of both experiments

Results of the first experiment

The analytical solution (equation (7.39)) used 90 terms for both the first and the middle series summation and the number of terms for the inner series was fixed according to the height of the electrodes. Figure 7.10 and figure 7.11 show the difference between the analytical solution and the experimental data in terms of magnitude and phase of the impedance. It can be seen from figure 7.10 and figure 7.11 that good agreement exists between the measured impedance and the computed impedance across the frequencies in terms of magnitude and phase.
Figure 7.10. Comparison between the computed magnitude and the measured magnitude

Figure 7.11. Comparison between the computed phase and the measured phase

Results of the second experiment

The analytical expression (equation (7.39)) used 90 terms for the first outer series, then 90 terms for the middle series and the number of terms for the inner series will be fixed
according to the height of the electrodes. Figure 7.12 and figure 7.13 show the results for the adjacent 1 in terms of magnitude and phase of the impedance.

Figure 7.12. Magnitude of the impedance for the adjacent 1 receive configuration

Figure 7.13. Phase of the impedance for the adjacent 1 receive configuration
Figure 7.14 and figure 7.15 show the results for the adjacent 2 in terms of magnitude and phase of the impedance.

Figure 7.14. Magnitude of the impedance for the adjacent 2 receive configuration

Figure 7.15. Phase of the impedance for the adjacent 2 receive configuration
Figure 7.16 and figure 7.17 show the results for the adjacent 3 in terms of magnitude and phase of the impedance.

Figure 7.16. Magnitude of the impedance for the adjacent 3 receive configuration

Figure 7.17. Phase of the impedance for the adjacent 3 receive configuration
7.2.4 Discussion

The first experiment shows that the analytical expression follows very well the experimental values both in terms of magnitude and in terms of phase at low frequency. Furthermore, the results were not normalized since direct comparisons could be done in terms of magnitude and phase.

In the second experiment, the data were normalized each time with the lowest frequency for a particular receive. It showed that there are few differences in terms of magnitude and phase. However, it must be noted that the Debye’s model employed does not necessarily accurately reflect the frequency dependencies of the red packed blood cells used in the practical experiment. There will also be noise coming from the hardware and also from the implementation of the analytical expression (i.e. convergence) for both experiments. Furthermore, the differences in the phase are predictable since already Alhargan (1993) reported to have similar differences when he compared his analytical expressions with real measurements. However, the phase seems to be good in the first experiment and also in the second experiment for adjacent 1.

Although further works must be performed in order to have more accurate results between the experimental results and the theoretical results, the next section will present a comparison between the new analytical expression (based on Helmholtz’s equation) and the analytical solution based on Poisson’s equation.

7.3 Determining the difference between Poisson’s equation and Helmholtz’s equation using a syringe filled of blood

In order to be consistent with the previous section, the same size of syringe filled with blood (packed red blood cells) exactly at the same level was modelled. In other words, a finite right circular cylinder having a height of 0.047 m, a radius of 0.015 m and on which two square electrodes of 0.005 m are attached in a polar configuration. In the
same plane of the injecting electrodes, seven point electrodes are attached forming with
the two injecting electrodes, eight receives. The current used in this experiment is of 0.1
mA.

The comparison will consists of comparing the boundary voltage profile collected from
the eight receives for the analytical expression based on Poisson’s equation and the new
analytical expression based on Helmholtz’s equation using frequency of 1 kHz, 10 kHz,
100 kHz and 1 MHz.

Figure 7.18 shows how the different adjacent receives are placed around the boundary.

![Figure 7.18. Eight adjacent receives placed around the boundary](image)

Figure 7.19 shows the boundary voltage profiles between Poisson’s equation and Helmholtz’s
equation at different frequencies for the eight receives.
Figure 7.19. Comparison between analytical expression based on Poisson’s equation and the new analytical expression based on Helmholtz’s equation

As the frequency increases, the voltages computed with the analytical expression based on Poisson’s equation become significantly different from the voltages computed with the new analytical expression based on the Helmholtz’s equation. This is especially true for receives close to the injected electrodes. For the measurements further away from the driving electrodes, this difference is smaller.

These differences are still under investigation and they may be due to the conductivity which was kept constant for Poisson’s equation for all the frequencies since the conductivity from the Debye’s dispersion model is effectively constant up to 1 MHz.

7.4 Discussion

In this section, the quasi-static assumptions have been applied to a number of biological tissues at different frequencies. It showed that these quasi-static conditions become
questionable around 1 MHz. This questions the validity of the governing equation used around and above 1 MHz. As this first study did not include any topology, it is important to find a way of assessing the difference between a Forward Problem which is based on quasi-static assumptions and the one which removes these assumptions. In order to derive a solution to the Forward Problem when non quasi-static conditions are used, a new equation must be derived from Maxwell’s equations. This section has shown that the resulting governing equation is the Helmholtz’s equation. The next step was to apply the same mathematical techniques used for deriving an analytical solution to the Forward Problem (for a finite right circular cylinder and for an elliptical cylinder) when quasi-static conditions are used, to the Helmholtz’s equation, thereby allowing a solution to the Forward Problem when non quasi-static conditions are applied to be derived. This section showed the derivation and the solution to this Forward Problem.

Experiments were conducted on a syringe filled of blood (packed red blood cells) in order to validate the new analytical expression based on Helmholtz’s equation. The experiments showed that the new analytical expression follows the same trend as the real measurements. Furthermore, good accuracy in terms of magnitude and phase were obtained with the first experiment. This was due to the fact that direct comparisons could be done. However, the second experiment involving the impulse system from Sheffield showed that the new expression followed the same trend as the real measurements. However, there were a slight difference in terms of magnitude and phase. Finally, the third theoretical experiment involved a direct comparison between the analytical expression based on Poisson’s equation and the new analytical expression based on Helmholtz’s equation. As the frequency increased, the differences appeared between both expression for receives close to the injected electrodes. Furthermore, differences existed among the computed voltage using the new analytical expression (based on Helmholtz’s equation) when the frequency was increased.

It can also be anticipated that differences between the analytical expression based on Poisson’s equation and the new expression based on Helmholtz’s equation will exist much earlier in the frequency for other biological material than blood since for instance,
the Debye dispersion for muscle possesses a dispersion much earlier in the frequency than for blood.

Although the results reported in this chapter are preliminary results, this chapter shows that attention must be said to which governing equation, namely Poisson (Laplace) or Helmholtz’s equation, should be used for the Forward Problem when the frequency is above 100 kHz. However, the exact frequency at which significant differences between these models will depend also on the biological material under investigation and the end application (static versus dynamic images). These results reveal that further investigation to these important issues is warranted.
Chapter 8
Future Work and Conclusion

This chapter will start by discussing the future work where it will suggest how the analytical solutions to the Forward Problem can be improved with particular attention to anisotropy and it will be shown how the work reported in this thesis could be extended to include anisotropic materials. Finally this chapter will conclude the thesis by summarizing the goals of the thesis.

8.1 Future Work

From what has been presented in the thesis, the future work should concentrate on five major points namely :

1) Modelling analytically the Forward Problem in three dimensions for different shapes,

2) Reconstruction of images using the elliptical cylinder,

3) Incorporating a better electrode model,

4) Modelling analytically the sensitivity matrix,

5) Modelling anisotropy and estimating the effect of incorporating anisotropy in images.

The first point should focus on using the same analytical tool presented in the thesis for deriving a full three dimensional analytical solution to the Forward Problem for a sphere and for an ellipsoid. It will involve the use of the spherical Bessel functions rather than Bessel functions. This future work could help with studies associated with imaging the head.
The second point is to extend the work presented on the elliptical cylinder in chapter 6 by reconstructing images for different eccentricities. It should determine how much difference in mapping the human thorax exists between images reconstructed using a finite circular cylinder and those reconstructed using an elliptical cylinder of different eccentricities.

The third point consists of improving the electrode model so far used in all the analytical solutions developed in this thesis. The electrode model employed is simple and some additional effort should be directed at improving the electrode models especially if static images are required. Perhaps one avenue to pursue in the future would be a FEM solution combined with an analytical solution in order to give enough accuracy around the electrodes using the analytical solution and enough speed for computing all the solution on the rest of the entire domain using the FEM solution. Therefore, high accuracy and speed could be combined.

By having modelled analytically the Forward Problem for the quasi-static (and non quasi-static) conditions and for two different geometrical shapes, namely a finite right circular cylinder and an elliptical cylinder, the fourth point should focus on the Inverse Problem by deriving a full three dimensional analytical sensitivity matrix for both geometrical shapes. Although the computation would be very slow, parallel machines could overcome this problem. By doing this for quasi-static and non quasi-static conditions, different sensitivity matrices could be built and their impact on the image reconstruction algorithms could be determined.

To date, EIT and MEIT assume that tissues are isotropic and, therefore, the conductivity and the permittivity distribution are the same in all directions. As a consequence, the Forward Problem and the Inverse Problem are simplified since the conductivity, the permittivity and the sources are only scalars. In reality, this is not true as tissues are anisotropic. If anisotropy has to be included, then this will be very challenging for EIT and MEIT since
1. A unique solution does not exist anymore (Lionheart, 1995).

2. The conductivity and the permittivity become tensors.

3. The electric field and the sources are vectors.

4. The Forward Problem and the Inverse Problem must be expressed in vector form.

However, it is possible to include anisotropy. The next section shows how anisotropy can be modelled based on the work presented in this thesis.

**Modelling Anisotropy**

In order to model anisotropy, the conductivity, permittivity and the permeability are tensor rather than scalar. Furthermore, the electric field and the sources and the solution to the Forward Problem will be vectors. Therefore, it is important to know how to extend the Green’s function to vector form. This can be achieved by the use of dyadic Green’s functions and they are presented in this section.

In chapter 4 and in this chapter, it was shown that the governing equation was either the Poisson’s equation or the Helmholtz’s equation given as

\[
\nabla^2 \Phi = -J(r_0) \text{ (Poisson’s equation)} \tag{8.1}
\]

\[
\nabla^2 \Phi + k^2 \Phi = -J(r_0) \text{ (Helmholtz’s equation)} \tag{8.2}
\]

The above equations can be expressed in one single equation as

\[
L \Phi = -J(r_0) \tag{8.3}
\]

where \(L\) is the operator defined to be either \(\nabla^2\) for Poisson’s equation (or Laplace’s equation) or to be \(\nabla^2 + k^2\) for Helmholtz’s equation.
A solution in the free-space to that kind of equation can be found as

\[ \Phi(r) = \int g(r|\mathbf{r}_0)J(\mathbf{r}_0)d\mathbf{v} \quad (8.4) \]

where \( g(r|\mathbf{r}_0) \) is the scalar Green’s function and \( J(\mathbf{r}_0) \) is the scalar source.

However, such a relationship would imply that the source is everywhere parallel, for instance to the x-axis, and therefore generate a field parallel to the same axis. This is not true when the medium is anisotropic. It is therefore necessary to use nine scalar Green’s functions to express the three components of \( \Phi(r) \) in terms of the three components of the source \( J(\mathbf{r}_0) \). Thus,

\[ \Phi_x(r) = \int \left[ g_x^x(r|\mathbf{r}_0)J_x(\mathbf{r}_0) + g_y^x(r|\mathbf{r}_0)J_y(\mathbf{r}_0) + g_z^x(r|\mathbf{r}_0)J_z(\mathbf{r}_0) \right] d\mathbf{v} \quad (8.5) \]
\[ \Phi_y(r) = \int \left[ g_x^y(r|\mathbf{r}_0)J_x(\mathbf{r}_0) + g_y^y(r|\mathbf{r}_0)J_y(\mathbf{r}_0) + g_z^y(r|\mathbf{r}_0)J_z(\mathbf{r}_0) \right] d\mathbf{v} \quad (8.6) \]
\[ \Phi_z(r) = \int \left[ g_x^z(r|\mathbf{r}_0)J_x(\mathbf{r}_0) + g_y^z(r|\mathbf{r}_0)J_y(\mathbf{r}_0) + g_z^z(r|\mathbf{r}_0)J_z(\mathbf{r}_0) \right] d\mathbf{v} \quad (8.7) \]

For instance, \( g_y^x(r|\mathbf{r}_0) \) measures the contribution of a y-oriented source, acting at \( r_0 \), to the x component of the field at \( r \). The above equation can be re-written as :

\[ \mathbf{\Phi}(r) = \int \left[ G_x(r|\mathbf{r}_0)J_x(\mathbf{r}_0) + G_y(r|\mathbf{r}_0)J_y(\mathbf{r}_0) + G_z(r|\mathbf{r}_0)J_z(\mathbf{r}_0) \right] d\mathbf{v} \quad (8.8) \]

where

\[ G_x(r|\mathbf{r}_0) = [g_x^x(r|\mathbf{r}_0), g_y^x(r|\mathbf{r}_0), g_z^x(r|\mathbf{r}_0)], \]
\[ G_y(r|\mathbf{r}_0) = [g_x^y(r|\mathbf{r}_0), g_y^y(r|\mathbf{r}_0), g_z^y(r|\mathbf{r}_0)], \]
\[ G_z(r|\mathbf{r}_0) = [g_x^z(r|\mathbf{r}_0), g_y^z(r|\mathbf{r}_0), g_z^z(r|\mathbf{r}_0)], \]
\[ \mathbf{\Phi}(r) = [\Phi_x(r), \Phi_y(r), \Phi_z(r)]. \]

Equation (8.8) can take a more optimized form called the *dyadic* form, expressed as :
\[
\Phi(r) = \int \overline{G}(r|r_0) \cdot J(r_0) dv
\]  
(8.9)

where
\[
\Phi(r) = [\Phi_x(r), \Phi_y(r), \Phi_z(r)] \text{ is the field vector,}
\]
\[
\overline{G}(r|r_0) = \begin{bmatrix}
g_x^x(r|r_0) & g_y^x(r|r_0) & g_z^x(r|r_0) 
g_x^y(r|r_0) & g_y^y(r|r_0) & g_z^y(r|r_0) 
g_x^z(r|r_0) & g_y^z(r|r_0) & g_z^z(r|r_0)
\end{bmatrix}
\]
is the dyadic Green’s function,
\[
J(r_0) = [J_x(r_0), J_y(r_0), J_z(r_0)] \text{ is the source vector.}
\]

By knowing the Green’s function to the scalar Poisson’s equation or Helmholtz’s equation for a precise geometrical shape such as the finite right circular cylinder given in chapter 4 or the elliptical cylinder given in chapter 6, the dyadic Green’s function can be constructed and a solution to the vector Forward Problem can be found. Such a solution to the Forward Problem will have the advantage of taking into account the anisotropy property of tissues since conductivity, permittivity and permeability will be tensors.

It can be seen that it is possible to construct a dyadic Green’s function from the scalar Green’s function. However, the disadvantage is that more computational time is required since the time used for computing one scalar Green’s function is multiplied by eight. Furthermore, it is expected that the dyadic Green’s function will have the same behaviour as the scalar Green’s function since the dyadic Green’s functions are constructed from them. Nevertheless by using a multiprocessor machine, it should be possible in theory to solve the vector Forward Problem. From that, a solution to the Inverse Problem could also be found and this will ensure future EIT and MEIT systems incorporate anisotropy. But this is beyond the scope of this thesis and is left open to future work. For more details on dyadic Green’s function, the reader is referred to the following references: (Collin, 1991), (Tai, 1993), (Wang, 1978), (Kleinermann and Avis, 1997).
8.2 Conclusion

The first aim of this thesis was to solve analytically the three dimensional Forward Problem for EIT. This has been achieved by the use of the Green’s function theory. It was shown that the analytical solution could be represented by two different forms. Results in the form of surface equipotentials, and equipotentials for internal planes have been presented for both forms. Whilst both forms of analytical expression reconstruct equipotentials, performance analysis of the convergence behaviour reveals that the convergence of the first form was good for points on the boundary and close to the boundary and was poor for points inside the circular cylinder. While conversely the convergence of the second form was good for points inside the circular cylinder and poor for points on the boundary. The first form was also less efficient in terms of computational time since it involves a triple series; while the second form only involves two series.

A problem associated with the use of small electrodes was discovered since the resulting equipotentials for both forms were distorted. This suggested that both forms have not yet reached convergence and therefore more summation terms in their series are required. In order to accelerate the convergence and to improve the performance when small electrodes are used, two accelerating techniques were introduced namely Euler and Epsilon. These two techniques not only improved the convergence, but also allowed the equipotentials to be correctly reconstructed when small electrodes were used. Furthermore, the first form was compared in terms of boundary voltage profile with other solvers namely, BEM, 1/r model and the Sheffield Mark III. It was concluded that the analytical solution presented in this thesis is accurate.

The second aim of this thesis was to reconstruct images using both forms presented in the first aim of the thesis. Both forms were able to reconstruct images of objects placed inside the circular cylinder using simulated data. It is still necessary to assess if they can reconstruct images with in-vivo data. However, the study performed by Metherall showed that the boundary voltage profile from the first form was more consistent with the
dataset collected using the Sheffield Mark III hardware than was the boundary voltage profile calculated using the 1/r model. Since Metherall was able to reconstruct three dimensional images from the human thorax using the 1/r model, it can be anticipated that the sensitivity computed from the analytical solution should also allow three dimensional images using real data to be reconstructed. Furthermore, the sensitivity matrix computed from the analytical solution would also be very important for IPT application since the models used in IPT are simple geometrical shapes. Whilst a finite right circular cylinder is sufficient as a model for Industrial Process Tomography, it may not be accurate enough as a model for the human thorax.

As a consequence, the third aim of the thesis was to investigate how analytical methods could be applied to more realistic models of the human thorax. It was shown how the Green’s function found for a finite right circular cylinder can be extended to an elliptical cylinder since an elliptical cylinder models the human thorax much more accurately by using the eccentricities. Again with the use of the Green’s function technique combined with the Eigenfunctions technique, an analytical solution to the Forward Problem was derived. It was shown that this analytical solution could reconstruct the equipotentials for different eccentricities. Furthermore, the potentials were compared with the potentials collected from a Boundary Element Method (BEM). They showed that they were in good agreement except near the electrode since the BEM models the electrodes in a different way than the analytical solution which uses a very simple electrode model. However, there is still a need for firstly studying the convergence when small electrodes are used and secondly to be able to reconstruct images using simulated data and real data. Unfortunately, due to the lack of EIT hardware, in-vitro validation studies were not performed.

The fourth aim of the thesis was to improve the modelling of EIT and MEIT. There are two ways of improving the modelling. The first way is to realise that the Forward Problem in EIT (and MEIT) assumes that the quasi-static conditions are valid. This has the advantage of simplifying Maxwell’s equations since they collapse to Poisson’s equation
Future Work and Conclusion

(or Laplace’s equation) with the assumption that the medium is isotropic and linear. As the frequency increases in MEIT, the quasi-static assumptions may no longer be valid. Recently several groups have increased the frequency above 1 MHz and therefore, these assumptions became more and more questionable. It is important to know when these assumptions may not be valid and to understand the effects of not including these assumptions in the model. This is the reason why preliminary work on how to model the Forward Problem when the quasi-static conditions are not applied has been presented in the thesis where a full analytical solution to the Forward Problem for a finite right circular cylinder when the quasi-static conditions are not applied has been derived. Some in-vitro validation of these mathematical expressions have been performed with promising results.

Finally, this thesis demonstrates that it is possible to derive three dimensional analytical solutions to the Forward Problem. The technique used for deriving the analytical solutions is very important. In this thesis, the technique of the Green’s functions combined with the Eigenfunction Expansion technique describes that it was possible to derive a three dimensional analytical solution, to extend it to N electrodes and to have different forms of the analytical solution allowing the problems of convergence to be tackled by different approaches. It also shows how quickly an analytical solution can be found for different topologies and for different constraints like, in this thesis, for the Multi-frequency EIT.

Having found analytical solutions to the Forward Problem for different topologies (finite right circular cylinder and elliptical cylinder), this thesis concludes by describing how anisotropy could be included. This will make the solution to the Forward Problem closer to reality and perhaps allow iterative algorithms to be more accurate since their first iteration will be closer to reality and therefore their convergence should be improved. This would be especially true for industrial process tomography since the shape of the object being imaged is a simple geometrical shape and the position of the electrodes does not change.
Therefore, this thesis contributes significantly to EIT and MEIT and I look forward to seeing this work being used in future.
Appendix A

Mittag-Leffler’s Expansion Theorem

By knowing that \( k^2 = -\beta_r^2 \) which means \( k = i\beta_r \) and defining

\[
F_n(i\beta_r) = \frac{J_n(i\beta_r\rho)J_n(i\beta_r\rho_0, i\beta_r a)}{4J_r(i\beta_r a)} - \frac{\delta_{0n}}{\pi\beta_r a^2}
\]  \( \text{(A.1)} \)

Mittag-Leffler’s expansion theorem can prove that

\[
\frac{J_n(i\beta_r\rho)J_n(i\beta_r\rho_0, i\beta_r a)}{4J_r(i\beta_r a)} - \frac{\delta_{0n}}{\pi\beta_r a^2} = \sum_{m=1}^{\infty} \frac{k_{nm}^2 J_n(k_{nm}\rho)J_n(k_{nm}\rho_0)}{\pi(k_{nm}^2 + \beta_r^2)(k_{nm}^2 a^2 - n^2)J_n(k_{nm} a)}
\]  \( \text{(A.2)} \)

where \( f_n(\rho, \rho_0) = Y_n^\ell(\rho_0)J_n(\rho) - J_n^\ell(\rho_0)Y_n(\rho) \)

Using Mittag-Leffler’s expansion theorem which states

\[
F(z) = F(0) + \sum_{m=1}^{\infty} R_m \left( \frac{1}{\alpha - p_m} + \frac{1}{p_m} \right)
\]  \( \text{(A.3)} \)

where \( P_m \) is the \( m^{\text{th}} \) pole of \( F(z) \) and \( R_m \) is the residue of \( F(z) \) at the \( m^{\text{th}} \) pole.

\( F_z \) is poles at \( k = \pm k_{nm} \) and \( F_n(0) = 0 \). Therefore

\[
F_n(k) = F_n(0) + \sum_{m=1}^{\infty} R_m \left( \frac{1}{k - k_{nm}} + \frac{1}{k_{nm}} \right) + \left( \frac{1}{k + k_{nm}} + \frac{1}{k_{nm}} \right)
\]

\[
F_n(k) = F_n(0) + \sum_{m=1}^{\infty} R_m \left( \frac{2k}{k^2 - k_{nm}^2} \right)
\]  \( \text{(A.4)} \)

where the residue is:

\[
R_m = \lim_{k \to k_{nm}} \frac{J_n(i\beta_r\rho)J_n(i\beta_r\rho_0, i\beta_r a)(k - k_{nm})}{4J_n(i\beta_r a)} - \frac{\delta_{0n}(k - k_{nm})}{\pi\beta_r a^2}
\]

\[
R_m = \frac{J_n(i\beta_r\rho)J_n(i\beta_r\rho_0, i\beta_r a)}{4aJ_n(i\beta_r a)}
\]  \( \text{(A.5)} \)

However,
\[ J_n'(a\beta_r i) = \frac{J_n(ai\beta_r)}{a^2} \left[ \frac{n^2}{k_{nm}^2} - a^2 \right] \quad (A.6) \]

By replacing \( i\beta_r = k_{nm} \) and by using these two relations:

\[
R_m = -\frac{k_{nm}^2 J_n(\rho_0 k_{nm}) J_n(\rho k_{nm})}{2\pi i \beta_r J_n^2(k_{nm}a)(k_{nm}a^2 - n^2)} \quad (A.7)
\]

Replacing \( k \) by \( i\beta_r \) and replacing \( R_m \) in equation (A.6):

\[
F_n(k) = \sum_{m=1}^{\infty} \frac{k_{nm}^2 J_n(\rho k_{nm}) J_n(\rho_0 k_{nm})}{\pi J_n^2(k_{nm}a)(k_{nm}a^2 - n^2)(\beta_r^2 + k_{nm}^2)} \quad (A.8)
\]
Appendix B

Derivation of the second form for a finite right circular cylinder

In chapter four the Green’s function of the first form was given as

\[
G(\rho, \phi, z \mid \rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_r \sigma_n J_n(k_{mn}\rho_0)J_n(k_{mn}\rho)}{2\pi \alpha^2 (k_{nm}^2 - \alpha^2)J_n^2(k_{mn}\alpha)} \times \frac{k_{nm}^2}{k_{nm}^2 + \beta_r^2} \cos(n(\phi - \phi_0)) \\
\times \cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c)) + \sum_{r=1}^{\infty} \frac{\cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c))}{c\beta_r^2 \alpha^2}
\]

(B.1)

where

- \(J_n\) : \(n^{th}\) Bessel function of the first kind,
- \(k_{nm}\) : \(m^{th}\) root of \(J'_n(k_{mn}\alpha) = 0\),
- \(\alpha\) : radius of the cylinder,
- \(c\) : half-length of the cylinder,
- \(\rho_0, \phi_0, z_0\) : position of the source,
- \(\rho, \phi, z\) : position of the computed potential,
- \(\beta_r = \frac{\pi r}{2c}\).

By using the Mittag-Leffler’s expansion theorem (presented in Appendix A), it can be shown that

\[
\frac{J_n(i\beta_r \rho)J_n(i\beta_r \rho_0, i\beta_r \alpha)}{4J'_n(i\beta_r \alpha)} = \frac{\delta_{0n}}{\pi \beta_r^2 \alpha^2} = \sum_{m=1}^{\infty} \frac{k_{nm}^2 J_n(k_{nm}\rho_0)J_n(k_{nm}\rho)}{\pi (k_{nm}^2 + \beta_r^2)(k_{nm}^2 \alpha^2 - \alpha^2)J_n^2(k_{nm}\alpha)}
\]

(B.2)

where

- \(f_n(\rho, a) = Y_n(a)J_n(\rho) - J'_n(a)Y_n(\rho)\),
- \(J_n\) : \(n^{th}\) Bessel function of the first kind,
- \(J'_n\) : first derivation of the \(n^{th}\) Bessel function of the first kind,
- \(Y_n\) : \(n^{th}\) Bessel function of the second kind,
- \(Y'_n\) : first derivation of \(n^{th}\) Bessel function of the second kind,
- \(\alpha\) : radius of the circular cylinder,
- \(\delta_{0n} = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0 \end{cases}\) (Kronecker symbol).
Factoring $k_{nm}$ expressions in equation (B.1),

$$G(\rho, \phi, z|\rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r \cos(n(\phi - \phi_0)) \cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))}{2c}$$

$$\times \sum_{m=1}^{\infty} \frac{k_{nm}^2 J_n(k_{nm} \rho) J_n(k_{nm} \rho_0)}{(k_{nm}^2 + \beta_r^2 r^2)(k_{nm}^2 a^2 - n^2) J_n^2(k_{nm} a)}$$

$$+ \sum_{r=1}^{\infty} \frac{\cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))}{c \pi \beta_r^2 a^2}$$  \hspace{1cm} (B.3)

Substituting equation (B.2) into equation (B.3), equation (B.3) becomes

$$G(\rho, \phi, z|\rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\sigma_n \sigma_r \cos(n(\phi - \phi_0)) \cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))}{2c}$$

$$\times \left\{ \frac{J_n(i \beta_r \rho) f_n(i \beta_r \rho_0, i \beta_r a)}{4 J_n(i \beta_r a)} - \frac{\delta_{0n}}{\pi \beta_r^2 a^2} \right\}$$

$$+ \sum_{r=1}^{\infty} \frac{\cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))}{c \pi \beta_r^2 a^2}$$  \hspace{1cm} (B.4)

$$G(\rho, \phi, z|\rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{\sigma_n \sigma_r \cos(n(\phi - \phi_0)) \cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))}{2c}$$

$$\times \left\{ \frac{J_n(i \beta_r \rho) f_n(i \beta_r \rho_0, i \beta_r a)}{4 J_n(i \beta_r a)} - \frac{\delta_{0n}}{\pi \beta_r^2 a^2} \right\}$$

$$+ \sum_{n=0}^{\infty} \sigma_n \sigma_0 \cos(n(\phi - \phi_0)) \left\{ \frac{J_n(i \beta_0 \rho) f_n(i \beta_0 \rho_0, i \beta_0 a)}{4 J_n(i \beta_0 a)} \right\}$$

$$- \frac{\delta_{0n}}{\pi \beta_0^2 a^2} \right\} + \sum_{r=1}^{\infty} \frac{\cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))}{c \pi \beta_r^2 a^2}$$  \hspace{1cm} (B.5)

$$G(\rho, \phi, z|\rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{\sigma_n J_n(i \beta_r \rho) f_n(i \beta_r \rho_0, i \beta_r a) \cos(n(\phi - \phi_0))}{4c J_n(i \beta_r a)}$$

$$\times \cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))$$

$$- \sum_{r=1}^{\infty} \frac{\cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))}{c \pi \beta_r^2 a^2}$$

$$+ \sum_{n=0}^{\infty} \sigma_n \sigma_0 \cos(n(\phi - \phi_0)) \left\{ \frac{J_n(i \beta_0 \rho) f_n(i \beta_0 \rho_0, i \beta_0 a)}{4 J_n(i \beta_0 a)} \right\}$$

$$- \frac{\delta_{0n}}{\pi \beta_0^2 a^2} \right\} + \sum_{r=1}^{\infty} \frac{\cos(\beta_r(z-c)) \cos(\beta_r(z_0-c))}{c \pi \beta_r^2 a^2}$$  \hspace{1cm} (B.6)
Finally, the Green’s function for the second form is:

\[
G(\rho, \phi, z | \rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{\beta_r = -1}^{1} \frac{\sigma_n J_n(i\beta_r \rho) f_n(i\beta_r \rho_0, i\beta_r a) \cos(n(\phi - \phi_0))}{4cJ_n'(i\beta_r a)} \\
\times \cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c)) + \frac{J_0(i\beta_0 \rho) f_0(i\beta_0 \rho_0, i\beta_0 a)}{8cJ_0'(i\beta_0 a)} \\
- \frac{1}{2\pi \beta_0^2 a^2} \sum_{n=1}^{\infty} \frac{J_n(i\beta_0 \rho) f_n(i\beta_0 \rho_0, i\beta_0 a)}{4cJ_n'(i\beta_0 a)} \\
\times \cos(n(\phi - \phi_0))
\]  

(B.7)

The above expression can be reduced for special cases using the following expressions:

\[
Y_n(x) \approx -\frac{n!2^n}{n\pi x^n}, \quad Y'_n(x) \approx \frac{n!2^n}{\pi x^{n+1}}, \quad J_n(x) \approx \frac{x^n}{2^n n!}, \quad J'_n(x) \approx \frac{x^{n-1}}{2^{n-1} (n-1)!} \quad \text{for } n = 0
\]

\[
\left| \frac{1}{x} \right| > |\ln(x)| \quad \text{for } 0 \leq x \leq 1
\]

for \(\beta_0 \to 0\), the following expressions hold:

\[
\frac{J_n(i\beta_0 \rho) f_n(i\beta_0 \rho_0, i\beta_0 a)}{J_n'(i\beta_0 a)} - \frac{1}{\pi \beta_0^2 a^2} = \frac{\pi \beta_0^2 a^2 J_0(i\beta_0 \rho) f_0(i\beta_0 \rho_0, i\beta_0 a) - 4J_0'(i\beta_0 a)}{4J_0'(i\beta_0 a)\pi \beta_0^2 a^2}
\]

\[
\rightarrow -\frac{1}{8\pi}
\]

(B.8)

Using the above relationships, equation (B.7) becomes:

\[
G(\rho, \phi, z | \rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{\beta_r = -1}^{1} \frac{\sigma_n J_n(i\beta_r \rho) f_n(i\beta_r \rho_0, i\beta_r a) \cos(n(\phi - \phi_0))}{4cJ_n'(i\beta_r a)} \\
\times \cos(\beta_r(z - c)) \cos(\beta_r(z_0 - c)) \cdot \frac{1}{16c\pi} \\
+ \sum_{n=1}^{\infty} \cos(n(\phi - \phi_0)) \left\{ \left( \frac{\rho \rho_0}{a^2} \right)^n + \left( \frac{\rho}{\rho_0} \right)^n \right\}
\]

(B.10)

For the special case where \(\rho_0 = a\), equation (B.10) becomes:
### Appendix B Derivation of the second form for a finite right circular cylinder

Equation (B.11) represents the Green’s function for the second form.

\[
G(\rho, \phi, z|\rho_0, \phi_0, z_0) = \sum_{n=0}^{\infty} \sum_{r=1}^{\infty} \frac{\sigma_n J_n(i\beta_r \rho) \cos(n(\phi - \phi_0)) \cos(\beta_r(z - c))}{2\pi aci\beta_r J'_n(i\beta_r a)} \\
\times \cos(\beta_r(z_0 - c)) - \frac{1}{16c\pi} \\
+ \sum_{r=1}^{\infty} \left(\frac{\rho}{a}\right)^n \frac{\cos(n(\phi - \phi_0))}{2cn\pi} 
\]

(B.11)
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